IV Caucasus Mathematic Olympiad
Maykop, March 15–20, 2019 year

Juniors. First day

March 16

1. In the kindergarten there is a big box with balls of three colors: red, blue and green, 100 balls in total. Once Pasha took out of the box 30 red, 10 blue, and 20 green balls and played with them. Then he lost five balls and returned the others back into the box. The next day, Sasha took out of the box 8 red, 18 blue, and 48 green balls. Is it possible to determine the color of at least one lost ball?

2. Determine if there exist five consecutive positive integers such that their LCM is a perfect square.

3. Points A' and B' lie inside the parallelogram $ABCD$ and points C' and D' lie outside of it, so that all sides of 8-gon $AA'BB'CC'DD'$ are equal. Prove that A', B', C', D' are concyclic.

4. Vova has a square grid 72×72. Unfortunately, n cells are stained with coffee. Determine if Vova always can cut out a clean square 3×3 without its central cell, if

(a) $n = 699$;

(b) $n = 750$.

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Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

Juniors. Second day

March 17

5. Vasya has a numeric expression

$$\square \cdot \square + \square \cdot \square$$

and 4 cards with numbers that can be put on 4 free places in the expression. Vasya tried to put cards in all possible ways and all the time obtained the same value as a result. Prove that equal numbers are written on three of his cards.

6. In a triangle ABC with $\angle BAC = 90^\circ$ let BL be the bisector, $L \in AC$. Let D be a point symmetrical to A with respect to BL. Let M be the circumcenter of ADC. Prove that CM, DL, and AB are concurrent.

7. 15 boxes are given. They all are initially empty. By one move it is allowed to choose some boxes and to put in them numbers of abricots which are pairwise distinct powers of 2. Find the least positive integer k such that it is possible to have equal numbers of abricots in all the boxes after k moves.

8. Determine if there exist positive integers $a_1, a_2, ..., a_{10}$, $b_1, b_2, ..., b_{10}$ satisfying the following property: for each non-empty subset S of $\{1, 2, \ldots, 10\}$ the sum $\sum\limits_{i \in S} a_i$ divides $\left(12 + \sum\limits_{i \in S} b_i\right)$.

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Seniors. First day

March 16

1. Pasha placed numbers from 1 to 100 in the cells of the square 10×10, each number exactly once. After that, Dima considered all sorts of squares, with the sides going along the grid lines, consisting of more than one cell, and painted in green the largest number in each such square (one number could be colored many times). Is it possible that all two-digit numbers are painted green?

2. In a triangle ABC let I be the incenter. Prove that the circle passing through A and touching BI at I, and the circle passing through B and touching AI at I, intersect at a point on the circumcircle of ABC.

3. Find all positive integers $n \geqslant 2$ such that there exists a permutation a_1, a_2, a_3, $\ldots$, a_{2n} of the numbers $1, 2, 3, \ldots, 2n$ satisfying

$$a_1 \cdot a_2 + a_3 \cdot a_4 + \ldots + a_{2n-3} \cdot a_{2n-2} = a_{2n-1} \cdot a_{2n}.$$

4. Dima has 100 rocks with pairwise distinct weights. He also has a strange pan scales: one should put exactly 10 rocks on each side. Call a pair of rocks *clear* if Dima can find out which of these two rocks is heavier. Find the least possible number of clear pairs.

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5. Given a triangle ABC with $BC = a$, $CA = b$, $AB = c$, $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$. Prove that

$$a\sin(\beta-\gamma) + b\sin(\gamma-\alpha) + c\sin(\alpha-\beta) = 0.$$

6. 15 boxes are given. They all are initially empty. By one move it is allowed to choose some boxes and to put in them numbers of abricots which are pairwise distinct powers of 2. Find the least positive integer k such that it is possible to have equal numbers of abricots in all the boxes after k moves.

7. On sides BC, CA, AB of a triangle ABC points K, L, M are chosen, respectively, and a point P is inside ABC is chosen so that $PL \parallel BC$, $PM \parallel CA$, $PK \parallel AB$. Determine if it is possible that each of three trapezoids $AMPL$, $BKPM$, $CLPK$ has an inscribed circle.

8. Determine if there exist pairwise distinct positive integers $a_1, a_2, \ldots, a_{101}$, b_1, b_2, $\ldots$, b_{101} satisfying the following property: for each non-empty subset S of $\{1, 2, \ldots, 101\}$ the sum $\sum\limits_{i\in S} a_i$ divides $\left(100! + \sum\limits_{i\in S} b_i\right)$.