IV Caucasus Mathematic Olympiad

Solutions
Day 1

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

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Adygea

1. Juniors

March 16

1. In the kindergarten there is a big box with balls of three colors: red, blue and green, 100 balls in total. Once Pasha took out of the box 30 red, 10 blue, and 20 green balls and played with them. Then he lost five balls and returned the others back into the box. The next day, Sasha took out of the box 8 red, 18 blue, and 48 green balls. Is it possible to determine the color of at least one lost ball?

Belov Dmitry

Answer. Yes, it is: at least one lost ball is red. **Solution.** Sasha took 18 blue ans 48 green balls mean that at least $18 + 48 = 66$ non red balls left in the box. If no red ball lost, there are at least 30 red balls in the box, so there are at least $66 + 30 = 96$ balls left in the box in total, which is incorrect. That mean Pasha lost at least one red ball.

2. Determine if there exist five consecutive positive integers such that their LCM is a perfect square.

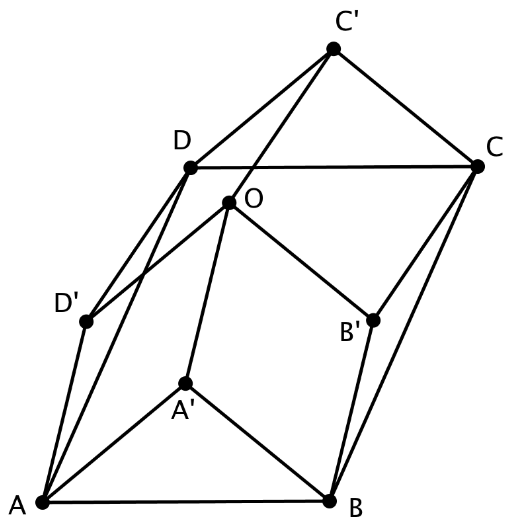
Kozhevnikov Pavel

Answer. No. **Solution.** Assume the contrary: $N = \mathrm{LCM}\,(n, n+1, n+2, n+3, n+4)$ is a persect square. Hence each prime has an even exponent in decomposition of N. Each prime $p \geqslant 5$ divides no more than one of the numbers $n, n+1, n+2, n+3, n+4$. Therefore, each $p \geqslant 5$ has an even (perhaps, 0) exponent in decomposition of each given number. $p = 2$ divides no more than three of given numbers, and has an odd exponent in no more than two given numbers. Similarly, $p = 3$ has an odd exponent in no more than one given numbers. Thus no more than three of given numbers have decomposition containing at least one odd exponent of some prime. Hence at least two given numbers contain only even exponents of primes in their decompositions, i.e. at least two given numbers are perfect squares.

Let $x^2 < y^2$ be these perfect squares. Hence $4 \geqslant y^2 - x^2 = (y - x)(y + x)$ which is true for $x = 1$ and $y = 2$ only. The only remaining possible case is $\text{НОК}(1, 2, 3, 4, 5) = 60$ not a perfect square.

3. Points A' and B' lie inside the parallelogram $ABCD$ and points C' and D' lie outside of it, so that all sides of 8-gon $AA'BB'CC'DD'$ are equal. Prove that A', B', C', D' are concyclic.

Bakaev Egor



Solution. By a denote the sidelength of the given 8-gon. Note that $\triangle AD'D$ is the shift of $\triangle BB'C$ by the vector $\overrightarrow{BA} = \overrightarrow{CD}$. Hence

$ABB'D'$ is a parallelogram. Similarly, $\triangle DC'C$ is the shift of $\triangle AA'B$ by $\overrightarrow{AD} = \overrightarrow{BC}$. Hence $BA'C'C$ is a parallelogram.

Take O such that $A'BB'O$ is rhombus, so that $OA' = OB' = a$. Since $BB' = A'O = AD'$, and $BB' \parallel A'O \parallel AD'$, it follows that $AA'OD'$ is a parallelogram, therefore $OD' = A'A = a$. Similarly we show that $OC' = a$. Thus O is at distance a from each of points A', B', C', D'.

4. Vova has a square grid 72×72. Unfortunately, n cells are stained with coffee. Determine if Vova always can cut out a clean square 3×3 without its central cell, if

(a) $n = 699$;

(b) $n = 750$.

Belov Dmitry

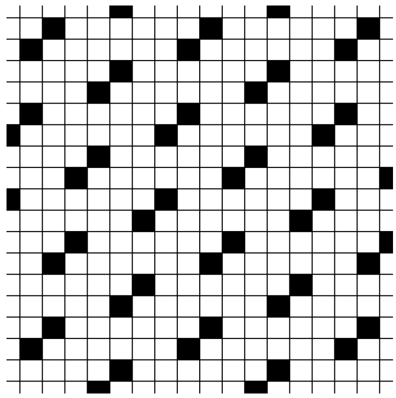
(a) Answer. Yes, he can. **Solution.** Each 3×3 square could be uniquely set by the Central cell. This could be chosen from inner square 70×70, i.e. $70\cdot70 = 4900$ ways. Therefore, the are 4900 3×3 square without Central cell. Let's prove that not all of them are stained.

Suppose the opposite. We will say that a dirty cell *prohibits* a 3×3 square without a Central cell if that dirty cell is one of the 8 cells of that square.

We divide all the stained cells into groups, combining into one group the stained cells, between which there is a path through the stained cells with transitions to neighboring cells on the side or diagonally. Consider one group s. Note that if there is only one cell C in s, then this cell C necessarily adjoins the border square 72×72, otherwise it would be possible to cut a square 3×3 without the Central cell C. In this case, a single cell of the group s prohibits no more than three squares 3×3 without a Central cell.

Now consider a group of stained cells T, consisting of more than one cell. Let's choose one of the cells of this group a. Choose also the side-adjacent or diagonal with a stained cell b. Together, the cells a and b prohibit no more than 14 squares 3×3. We will add the cells of T group to the already selected ones, each time adding a cell that is adjacent to one of the already selected ones. Then each added cell prohibits no more than 6 of new squares 3×3, which we have not yet considered as forbidden by the group of cells T. Therefore, if there are only k cells in the t group, then the number of 3×3 squares they prohibit does not exceed $7k$.

Thus, 699 cells in total prohibit no more than $699 \cdot 7 = 4893$ squares $3\times$ 3 without a Central cell. But there are 4900 such squares. So, there is a clean square 3×3 without a Central cell, which Vova will be able to cut.



(b) Answer. No, not always. **Solution.** Color a plane as shown in the picture. Let black color sells mean the celss soiled with coffee. Note that it is impossible to cut a square 3×3 without a Central cell from such the plane.

Place the square 72×72 on this plane so that its lower left corner square 2×2 does not contain black cells. We will show that black cells in a square will be no more than 750, then this example will fit the conditions of the problem, and Vova will not be able to cut out a square 3×3 without a Central cell from it.

Select the first two columns in the square. We divide them to the lower square 2×2, in which there are no black cells, and 10 vertical strips 1×14, each of which has 2 black cells. There are 20 black cells in total. The rest of the board can be divided into horizontal strips 7×1. There are $72 \cdot 10 = 720$ pieces, and each strip has exactly one black cell. So, there are exactly 740 black cells in such 72×72 square, which is less than 750, and the example fits.

2. Seniors *March 16*

1. Pasha placed numbers from 1 to 100 in the cells of the square 10×10, each number exactly once. After that, Dima considered all sorts of squares, with the sides going along the grid lines, consisting of more than one cell, and painted in green the largest number in each such square (one number could be colored many times). Is it possible that all two-digit numbers are painted green?

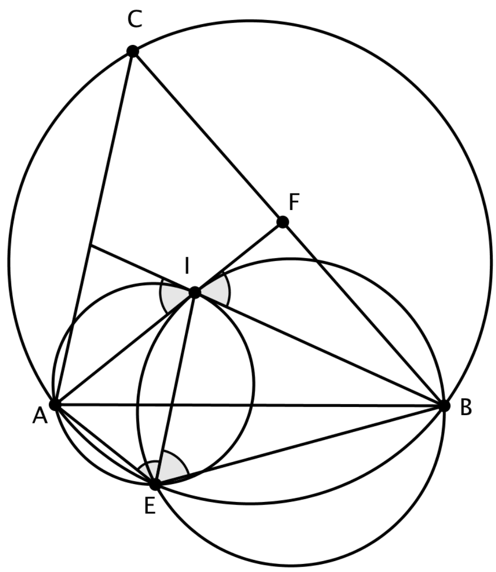
Bragin Vladimir

Answer. No, it is impossible. **Solution.** Consider arbitrary green number. It is the largest number in some square. In this square we can select sqruare 2×2 including any cell, particularly green number. The considered green number is also the largest in selected 2×2 square. Thus, any green number is the largest in some 2×2 square. But there are only 81 2×2 squares, becauce any 2×2 is determined by its left bottom cell. So there cannot be more than 81 green numbers, but there are 90 two-digit numbers.

2. In a triangle ABC let I be the incenter. Prove that the circle passing through A and touching BI at I, and the circle passing through B and touching AI at I, intersect at a point on the circumcircle of ABC.

Emelyanov Lev

Solution. Let E be the common point of given circles. We have $\angle BEI = \angle BIF$ (where F lies on the prolongation of AI beyond I), hence $\angle BEI = \angle BAI + \angle ABI = \frac{1}{2}(\angle A + \angle B)$. Similarly, $\angle AEI = \frac{1}{2}(\angle A + \angle B)$. Therefore, $\angle AEB = \angle A + \angle B = 180^\circ - \angle C$. This means that $ACBE$ is inscribed.



3. Find all positive integers $n \geqslant 2$ such that there exists a permutation $a_1, a_2, a_3, \ldots, a_{2n}$ of the numbers $1, 2, 3, \ldots, 2n$ satisfying

$$a_1 \cdot a_2 + a_3 \cdot a_4 + \ldots + a_{2n-3} \cdot a_{2n-2} = a_{2n-1} \cdot a_{2n}.$$

Saghafian Morteza

Answer. $n = 3, 4, 5, 6, 7$. **Solution.** It is obvious that the maximum of RHS is $2n(2n-1)$.

Statement. The minimum value of LHS is

$$1 \cdot (2n-2) + 2 \cdot (2n-3) + \ldots + (n-1) \cdot n.$$

We'll prove this statement later, first we shall use it to copmplete the solution The minimum sum in LHS can be rewtitten

$$\sum_{i=1}^{n-1} i(2n-1-i) = (2n-1)\sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} i^2 =$$
$$(2n-1)\frac{(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6} = \frac{n(n-1)(2n-1)}{3}.$$

So the following inequality holds

$$\frac{n(n-1)(2n-1)}{3} \leqslant 2n(2n-1).$$

From which we conclude $n - 1 \leqslant 6$, or $n \leqslant 7$.

It is clear that the required is impossible for $n = 2$.

It remains to show examples for $n = 3, 4, 5, 6, 7$. For $n = 7$ the minimum of LHS is equal to maximum of RHS:

$$1 \cdot 12 + 2 \cdot 11 + 3 \cdot 10 + 4 \cdot 9 + 5 \cdot 8 + 6 \cdot 7 = 13 \cdot 14.$$

For $n = 3$: $1 \cdot 2 + 3 \cdot 6 = 4 \cdot 5$.

For $n = 4$: $8 \cdot 5 = 7 \cdot 4 + 6 \cdot 1 + 2 \cdot 3$.

For $n = 5$: $8 \cdot 9 = 1 \cdot 10 + 2 \cdot 7 + 3 \cdot 6 + 4 \cdot 5$.

for $n = 6$: $12 \cdot 10 = 11 \cdot 1 + 9 \cdot 5 + 8 \cdot 3 + 7 \cdot 4 + 6 \cdot 2$.

Proof of the statement. Let us show, that if there are $a_1 < a_2 < \ldots < a_{2n}$, that the minimum value of sum of products in pairs equals to $a_1a_{2n} + a_2a_{2n-1} + \ldots + a_na_{n+1}$. We'll use induction.

Base of induction is the case $n = 2$. Let us prove, that if a is the least of 4 numbers and d is the largest, that $ad + bc < ab + cd$. We can rewrite it $a(d - b) < c(d - b)$ and this follows from $a < c$ and $d - b > 0$. The step of induction $n - 1 \to n$. Consider $2n$ numbers $a_1 < a_2 < \ldots < a_{2n}$ and the way of deviding into pairs with minimal sum of products. Form the base case we can conclude that a_1 must be in one pair with a_{2n}, otherwise . By induction hypothesis the least possible sum of products for numbers $a_2 < a_3 \ldots < a_{2n-1}$ $a_2a_{2n-1} + a_3a_{2n-2} + \ldots + a_na_{n+1}$.

4. Dima has 100 rocks with pairwise distinct weights. He also has a strange pan scales: one should put exactly 10 rocks on each side. Call a pair of rocks *clear* if Dima can find out which of these two rocks is heavier. Find the least possible number of clear pairs.

Bragin Vladimir

Answer. $\binom{100}{2} - \binom{19}{2} = 4779$. **Solution.** A pair of rocks is called *bad* if it is not clear.

Example. There are a set of 100 rocks containing a subset of 19 rocks such that any two rocks in the subset form a bad pair.

Let the weights of rocks be equal to 100, 200, 400, $\ldots$, $2^{80} \cdot 100$, $2^{100}+1$, $2^{100}+2$, $\ldots$, $2^{100} + 19$. A rock is of weight more than 2^{100} is called *heavy*, otherwise it is *light*. We claim that it is impossible to distinguish heavy rocks even if we weight all possible pairs of subsets of 10 rocks using our pan scale. The following obvious facts easily imply the claim.

Fact 1. If a set of 10 rocks has more heavy rocks than the second one, then it is also heavier than another.

Fact 2. If two sets of 10 rocks have the same number of heavy rocks, then the set with the heaviest rock among light ones (in these two sets) is heavier.

Proof of the upper bound. First, we need an auxallary definition: A *switching* of two stones A and B for a weighting of two sets of size 10 is the weighting of these sets after the replacement the stone A by B and the stone B by A, i.e. if A and B lie on the same side or do not lie on the pan scales, then we have the same weighting, if only one stone eighther A or B lies on the pan scales, then we just replace this stone by another and do not touch the rest rocks, if the stone A lies on one side and B lies on another, then A and B switch their sides and all the rest rocks lie on the same sides.

Let us show that there are at most 171 bad pairs of rocks. For that we need a few facts.

Fact 3. A pair $\{A, B\}$ of two rocks is bad iff the result of any weighting with A replaced by B and B replaced by A is the same.

See the proof of Fact 3 below.

Fact 4. If the pairs $\{X, Y\}$ and $\{X, Z\}$ are bad, then the pair $\{Y, Z\}$ is also bad.

Proof. Consider a sequence of weightings: any weighting ω_1, the switching ω_2 of X and Y for ω_1, the switching ω_3 of Z and X for ω_2, and then again the switching ω_4 of Y and X for ω_3. By Fact 3, the results of these weightings are the same. But ω_4 is the switching of Y and Z for ω_1. Indeed: $(X, Y, Z) \to (Y, X, Z) \to$ $\to (Y, Z, X) \to (X, Z, Y)$

This implies that the result of the switching of Z and Y for any weighting is the same. Thus, by Fact 3, the pair $\{Y, Z\}$ is bad. This completes the proof of the fact.

Consider a graph whose vertices are rocks and edges are pairs of bad rocks. By Fact 4, each component of the graph is a complete graph.

Claim 5. For any 20 rocks $A_1, A_2, \ldots, A_{20}$, among pairs $\{A_1, A_2\}$, $\{A_3, A_4\}$, $\ldots$, $\{A_{19}, A_{20}\}$ their is at least one clear pair.

Proof. Let us weight such sets of rocks: $\{A_1, A_3, \ldots, A_{19}\}$ and $\{A_2, A_4, \ldots, A_{20}\}$. If the pan scales shows the equilibrium then after a switching of any pair we easily understand which rock in the pair is heavier. If one set of 10 rocks is heavier than another, then we switch one by one our 10 pairs of rocks. After all changes the result of the last weighting is opposite. It means there is a switching (of some pair $\{A, B\}$) changing the result, and thus the pair $\{A, B\}$ is clear. This finishes the proof of Fact 5.

Now consider the largest matching of each component of our graph. Let the size of the matching of i-th component is n_i. According to Fact 5, the sum of n_i is at most 9. So, in the i-th component there is at most $2n_i + 1$ vertices, and that why there are at most $n_i(2n_i + 1)$ edges. Therefore,

$$\sum n_i(2n_i + 1) = 2\sum n_i^2 + \sum n_i \leqslant 2(\sum n_i)^2 + \sum n_i \leqslant 2 \cdot 9^2 + 9 = 171.$$

Proof of Fact 3. A proof of the sufficient implication is obvious. Thus, let us prove that if the result of the switching of a pair for a weighting is different from the result of the weighting, then this pair is clear.

There are several possibilities: each stone of the pair lies on the right side of the pan scales, on the left side or does not lie on the pan scales. If both rocks lie on the same side or do not lie on the pan scales, then the result is the same (since the weighting is the same).

If stones are on different side of the scale pan, then there are several possible cases:

$>, =$ In this case the rock on the left side is heavier.

$>,<$ In this case the rock on the left side is heavier.

$=,>$ In this case the rock on the right side is heavier.

$=,<$ In this case the rock on the left side is heavier.

$<,=$ In this case the rock on the right side is heavier.

$<,>$ In this case the rock on the right side is heavier.

If one of the stones does not lie on the scale pan, then the weight on one side does not change after the switching. Thus, we easily see which rock of the pair is heavier, and so this pair is clear.