V Caucasus Mathematic Olympiad

Solutions
Day 1

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

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March 13–18, 2020 year
Maykop
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1. Juniors *March 14*

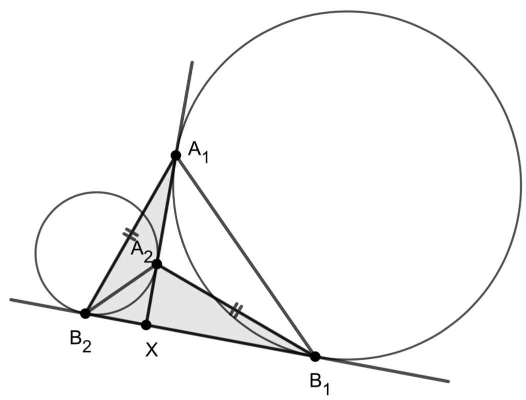
1. Using one magic nut, Wicked Witch can either turn a flea into a beetle or a spider into a bug; but using one magic acorn, she can either turn a flea into a spider or a beetle into a bug. In the evening Wicked Witch had spent 20 magic nuts and 23 magic acorns. By a sequence of these actions, the number of beetles increased by 5. Determine what was the change in the number of spiders.

E. Bakaev

Answer. The number of spiders increased by 8. **Solution.** When using a magic nut either the number b of bugs increases by 1 or the number s of spiders decreases by 1, so the difference $b - s$ increases by 1. When using a magic acorn vice versa: either the number b of bugs decreases by 1 or the number s of spiders increases by 1, so the difference $b-s$ decreases by 1. Due to these operations, the difference $b - s$ increased by 20 and decreased by 23, therefore in total it decreased by 3. Since b increased by 5, then s increased by $5 + 3 = 8$.

2. Let ω_1 and ω_2 be two non-intersecting circles. Let one of its internal tangents touch ω_1 and ω_2 at A_1 and A_2, respectively, and let one of its external tangents touch ω_1 and ω_2 at B_1 and B_2, respectively. Prove that if $A_1B_2 = A_2B_1$, then $A_1B_2 \perp A_2B_1$.

P. Kozhevnikov



Solution. Let X be the intersection of tangents. From equality of corresponding segments of tangents: $XA_1 = XB_1$ and $XA_2 = XB_2$. Since $A_1B_2 = A_2B_1$, triangles A_1XB_2 and B_1XA_2 are equal (by SSS). Therefore, $\angle A_1XB_2 = \angle A_2XB_1$, and both these angles are right angles.

Rotation by 90° with center X takes triangle A_1XB_2 to B_1XA_2, hence the corresponding sides of these triangles are perpendicular.

3. Let a_n be a sequence given by $a_1 = 18$, and $a_n = a_{n-1}^2 + 6a_{n-1}$, for $n > 1$. Prove that this sequence contains no perfect powers.

S. Luchinin

Solution. The number $18 = 2^1 \cdot 3^2$. We also calculate $a_2 = 18^2 + 6 \cdot 18 = 432 = = 2^4 \cdot 3^3$. We prove that every member of the sequence starting from the second one can be represented as $a_k = 2^{k+2} \cdot 3^{k+1} \cdot t_k$, where t_k is an integer not divisible by 2 and by 3.

In fact, this is true for a_2; we prove that if the statement is true for a_i, then it is also true for a_{i+1}. By assumption, $a_i = 2^{i+2} \cdot 3^{i+1} \cdot t_i$, where НОД$(t_i, 6) = 1$. Under the terms,

$$a_{i+1} = a_i^2 + 6a_i = a_i(a_i + 6) = 2^{i+3} \cdot 3^{i+2} \cdot t_i(2^{i+1} \cdot 3^i \cdot t_i + 1).$$

Since $i \geqslant 2$, the last multiplier is not divisible by either 2 or 3, the same is true for t_i. This means that the degrees of occurrence of 2 and 3 in the prime factorization increased by exactly 1.

It remains to note that a number of the form $2^{s+1} \cdot 3^s \cdot m$, where НОД$(m, 6) = 1$, cannot be a power (higher than one) of a positive integer number. Indeed, if it were p-th power of a positive integer number, then all prime factors would be included in it in a power multiple of p. Then both the numbers $s+1$ and s would have to be divisible by p, which is impossible for $p > 1$.

4. Positive integers $n, k > 1$ are given. Pasha and Vova play a game on a board $n \times k$. Pasha begins, and thereafter they alternate using the following moves. On each move a player should mark a border of length 1 between two adjacent cells. The player loses if, after his move, it is not possible to start from the bottom left cell and reach the top right cell moving through edges of adjacent cells without crossing any marked border. Determine which of the players has a winning strategy.

S. Luchinin

Answer. Vova. **Solution.** Consider the situation "for a move before losing", i.e. in which it is impossible to make a move without immidiate loss.

Since in this situation the game has not yet been lost by anyone, there is a path p through various cells $A = A_0, A_1, A_2, \ldots, A_t = B$, where A — the lower left cell, B — the upper right cell, and A_{i-1} and A_i — a pair of side-adjacent cells for each $i = 1, \ldots, t$. Note that on the t borders between the cells A_{i-1}, A_i there are no edges, and on all the other $m = n(k-1) + k(n-1) - t$ the borders must have sides, otherwise in this situation, a move could be made, after which the path p remains, in contradiction with the choice situations "for a move before losing". So, by this point, m moves had been made. We show that m is even. This will mean that in the situation "for a move before losing" can only be Pasha. This means that Vova always has a move that does not lead to an immediate loss, and since the game will end in a finite number of moves, Vova can win.

Color the cells of the Board in a staggered order. Cells A and B are monochrome if n and k are of the same parity, and multicolored otherwise. When you move along the path p, the color changes every time you move to the next cell. This means that the parity of the number of transitions required to get from A to B is the same as the parity of $n+k$. Then $t-(n+k)$ is even, so $m = n(k-1)+k(n-1)-t$ is also even, as required.

2. Сеньоры *14 марта*

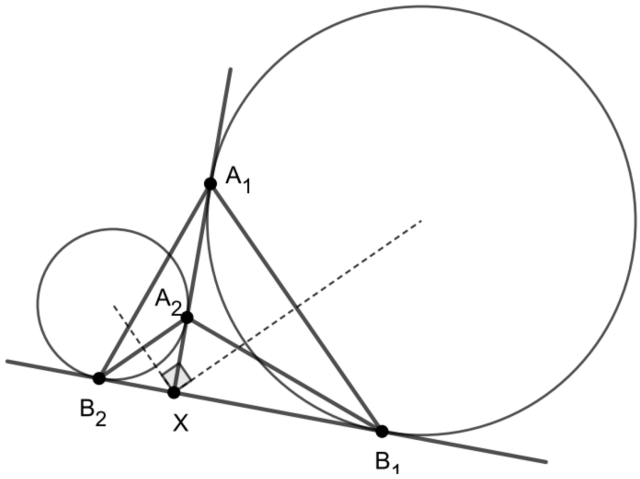
1. Determine if there exists a finite set A of positive integers satisfying the following condition: for each $a \in A$ at least one of two numbers $2a$ and $a/3$ belongs to A.

Seyyedsalehi M.H.

Answer. No. **Solution.** Assume that such A exists. Among elements of A choose numbers divisible by the maximal power of 2, and then choose the least number a among these numbers. Let a be divisible by 2^k but not by 2^{k+1}. Hence $2a$ is divisible by 2^{k+1}, and, by choice of a, $2a$ does not belong to A. This means that $a/3$ should be a positive integer and belong to A. In this case $a/3$ is divisible by 2^k. Since $a/3 < a$, this contradicts the definition of a.

2. Let ω_1 and ω_2 be two non-intersecting circles. Let one of its internal tangents touch ω_1 and ω_2 at A_1 and A_2, respectively, and let one of its external tangents touch ω_1 and ω_2 at B_1 and B_2, respectively. Prove that if $A_1B_2 \perp A_2B_1$, then $A_1B_2 = A_2B_1$.

P. Kozhevnikov



Solution. Let X be the intersection of tangents. A_1B_1 is perpendicular to the bisector of the angle A_1XB_1, while A_2B_2 is perpendicular to the bisector of the angle A_2XB_2, which is adjacent to the angle A_1XB_1. Since these bisectore are perpendicular, we have $A_1B_1 \perp A_2B_2$. From this and from the condition $A_1B_2 \perp A_2B_1$ it follows that A_1, A_2, B_1, B_2 is an orthocentic quadruple (in particular, A_2 is the orthocenter of $A_1B_1B_2$. Thus $A_1A_2 \perp B_1B_2$.

From equality of the corresponding segments of the tangents we have $XA_1 = XB_1$ and $XA_2 = XB_2$, hance right-angled triangles A_1XB_2 are B_1XA_2 equal. Thus $A_1B_2 = A_2B_1$, as required.

3. Peter and Basil play the following game on a horizontal table 1×2019. Initially Peter chooses n positive integers and writes them on a board. After that Basil puts a coin in one of the cells. Then at each move, Peter announces a number s among the numbers written on the board, and Basil needs to rearrange the coin by s cells, if it is possible: either to the left, or to the right, by his decision. In case it is not possible to rearrange the coin by s cells neither to the left, nor to the right, the coin stays in the current cell. Find the least n such that Peter can play so that the coin will visit all the cells, regardless of the way Basil plays.

M. Didin

Answer. $n = 2$. **Solution.** It is clear that $n = 1$ does not work: for $n = 1$ Peter needs to announce the same number on each move. Thus if Basil can shift the coin by one move from A to B, then further he can shift the coin from B to A, A to B, etc.

Let $n = 2$, and let Peter write numbers $k = 1009$ и $k + 1 = 1010$. Let us enumerate the cells with the numbers $-k, -k + 1, \dots, -2, -1, 0, 1, 2, \dots, k$.

Let Basil initially put the coin in the cell $a < 0$. Then let Peter announce k, and then $k + 1$. The responce of Basil is unique: he needs to shift the coin to the cell $a + k$, and then to the cell $a - 1$. Continue in the same manner, until the coin gets to the cell $-k$ (thus the coin visits cells a, $a + k$, $a - 1$, $a + k - 1$, $a - 2$, $a + k - 2, \dots, 2, -k + 1, 1, -k$). After that, let Peter perform $2k$ moves alternatively announcing $k + 1$ and k. The sequence of shifts is unique, and the coin visits cells $-k, 1, -k+1, 2, -k+2, \dots, k-1, -1, k, 0$. Using this strategy, Peter wins. Analogous strategy works ifthe number of the initial cell is greater than 0. Finally, if initially the coin is in the cell number 0, then let Peter on the first move announce k, so that Basil needs to move the coin to one of the cells $\pm k$. After that, Peter can apply the above strategy.

4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n the number $f(m) + n - m$ is divisible by $f(n)$.

M. Saghafian

Answer. 1) $f(n) = n + c$ for all c;
2) $f(n) = 1$;
3) $f(n) = 1$ if n is even and $f(n) = 2$ if n is odd;
4) $f(n) = 2$ if n is even and $f(n) = 1$ if n is odd.

Solution. Suppose $f(n)$ accepts infinitely many values. We can rewrite the condition $f(m) - m \equiv f(n) - n \pmod{f(n)}$. Suppose $f(a) - a \neq f(b) - b$ for some a and b. Consider n for which $f(n) > |f(a) - a| + |f(b) - b|$. Then $f(a) - a \equiv f(n) - n \pmod{f(n)}$ and $f(b) - b \equiv f(n) - n \pmod{f(n)}$. That means that $f(a) - a - (f(b) - b)$ is devisible by $f(n)$, but $f(n)$ is bigger than $|f(a) - a - (f(b) - b)|$ and we get a contradiction.

Hence if $f(n)$ accept infinitely many values, then $f(n) - n$ is constant. It is obvious all these functions are suitable.

Consider the case $f(n)$ accepts finitely many values. Rewrite the condition in a following way

$$m - f(m) \equiv n \pmod{f(n)}. \tag{1}$$

Denote s the maximum value of $f(n)$, denote a the corresponding arguement. This means $f(a) = s$.

Consider the case $s = 1$, then for every n, $f(n) = 1$.

Consider the case $s = 2$. Then if a is odd, then from (1) we get that m and $f(m)$ must be different oddity m. Hence $f(n) = 1$ for even n a $f(n) = 2$ for odd n. If a is even, then m and $f(m)$ must be the same oddity m, so $f(n) = 1$ for odd n and $f(n) = 2$ for even n.

Consider the case $s > 2$. Let us put a instead n into (1). Then we get $f(m) \equiv m - a \pmod{s}$. Moreover $1 \leqslant f(m) \leqslant s$, this means that for any k, $f(a + ks) = s$ and for any t $f(a + ts - 1) = s - 1$. Let us put) $a + ks$ instead m and $a + ts - 1$ instead n in (1, $a + ks - s \equiv a + ts - 1 \pmod{s - 1}$ r $1 \equiv t - k \pmod{s - 1}$. As k and t can be chosen arbitrary, we get a contradiction.