V Caucasus Mathematic Olympiad

Solutions
Day 2

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

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Maykop
Adygea

1. Juniors *March 15*

5. Find the number of pairs of positive integers a and b such that $a \leqslant 100\,000$, $b \leqslant 100\,000$, and

$$\frac{a^3 - b}{a^3 + b} = \frac{b^2 - a^2}{b^2 + a^2}.$$

D. Belov

Answer. 10 pairs. **Solution.** Convert this expression by multiplying fractions by the product of denominators:

$$a^3b^2 + a^5 - b^3 - a^2b = a^3b^2 - a^5 + b^3 - a^2b,$$

that after more converts takes the form $a^5 = b^3$. Since the transformations are equivalent (the denominators of the original fractions for natural numbers are non-zero), it is sufficient to find the number of pairs of natural a and b for which $a^5 = b^3$. Under this condition, b is the fifth power, and the number a is an exact cube. There are 10 fifth degrees up to $100\,000$ in total, and each will correspond to a cube that does not exceed $100\,000$. So, there are exactly 10 suitable pairs.

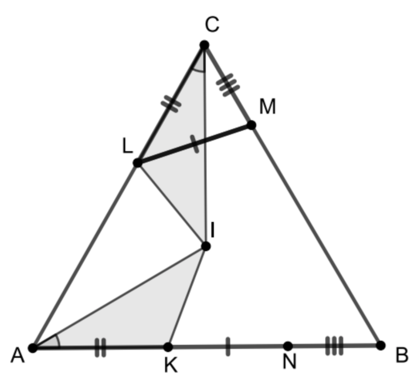
6. All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

E. Bakaev

Solution. Let's we fit this polygon into the circle. Consider each of the 50 diameters formed by the vertices of the polygon. Note that if there are two diameters among them for which the vertex color set matches, then the four ends of these diameters form a rectangle that matches the condition. There are $10 \cdot 9 : 2 = 45$ options to paint the ends of the diameter in two different colors (without taking into account the order). This means that if a suitable rectangle has not yet been found, there will be at least five diameters whose ends are of the same color. Choose any two, and their ends form a suitable rectangle.

7. An equilateral triangle ABC is given. Points K and N lie in the segment AB, a point L lies in the segment AC, and a point M lies in the segment BC so that $CL = AK$, $CM = BN$, $ML = KN$. Prove that $KL \parallel MN$.

E. Bakaev

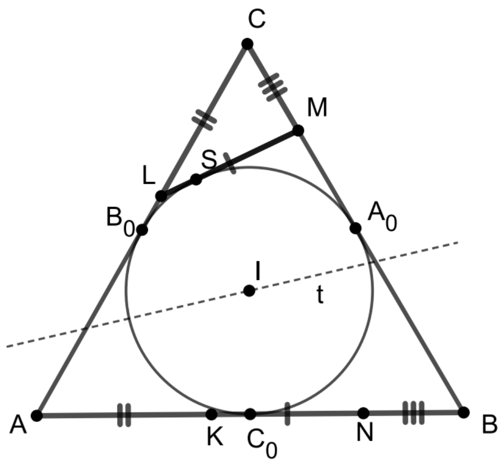


Solution. First let us note that A, K, N, B are placed on AB in this order. Otherwise we have $LM = KN < AK = CL$, and similarly, $LM < CM$. Thus in triangle CLM

the side LM is minimal, which is impossible, since it is opposite to the angle 60°, which is not minimal.

Solution 1. Let I be the incenter of ABC. Since $IC = IA$, $\angle ICL = \angle IAK = 30^\circ$, and $CL = AK$, triangles ICL and IAK are equal. Hence $IL = IK$, and similarly, $IM = IN$.

Thus triangles ILM and IKN are equal by SSS, hence they are symmetric to each other in some line ℓ, passing through I. Hence each of lines KL, MN is perpendicular to ℓ.



Solution 2. Let the incircle ω touch the sides AB, BC, CA at C_0, A_0, B_0, respectively, hence C_0, A_0, B_0 are the midpoints. Perimeter of CLM equals $AB = AC = BC$, hence its semiperimeter equals $CA_0 = CB_0$. It follows that ω is the excircle of CLM. Both lines KN and LM are tangents to ω, they are symmetric to each other in some line t passing through I.

Let S be the touching point of LM and ω, so that S and C_0 are symmetric in t. We have $LS = B_0L = B_0C - LC = C_0A - KA = KC_0$. We obtain the equality which means that that LS and KC_0 are symmetric in t, hence $KL \perp t$. Similarly, $MN \perp t$.

8. Let the real numbers a, b, and c satisfy

$$abc + a + b + c = ab + bc + ca + 5.$$

Find the minimal possible value of $a^2 + b^2 + c^2$.

A. Antropov, V. Bragin

Answer. 6. **Solution.** Transform the condition as $(a-1)(b-1)(c-1) = 4$. Let's immediately give an example when $a = -1$, $b = -1$, $c = 2$, then $a^2+b^2+c^2 = 6$. We prove that this value is minimal. There are two possible cases.

Case 1. All brackets are positive. Then all three numbers a, b and c are greater than one. Since the product of three positive brackets is 4, at least one bracket is greater than 1, i.e. at least one of the numbers a, b, and c is greater than 2. Then the sum of three squares is greater than $1^2 + 1^2 + 2^2 = 6$, and the value found above is exactly 6.

Case 2. Two brackets are negative and one bracket is positive. Let the brackets $a - 1$ and $b - 1$be negative. Replace them with $1 - a$ and $1 - b$ so that all three multipliers become positive. Also multiply the product by 2 to get

$$(1 - a)(1 - b)(2c - 2) = 8.$$

For the numbers $1-a$, $1-b$, and $2-2c$, we apply the inequality between the arithmetic and geometric averages for three numbers, then

$$\frac{(1-a)+(1-b)+(2c-2)}{3} \geqslant \sqrt[3]{(1-a)(1-b)(2c-2)} = 2,$$

$$2c-a-b \geqslant 6.$$

We prove that the last condition implies that $a^2+b^2+c^2 \geqslant 6$. Note that three conditions are met: $a^2+1 \geqslant -2a$, $b^2+1 \geqslant -2b$, $c^2+4 \geqslant 4c$. Add these three inequalities and we get

$$a^2+b^2+c^2+6 \geqslant 4c-2a-2b \geqslant 12,$$

where the last inequality is obtained by multiplying the condition $2c-a-b \geqslant 6$ by 2. So, $a^2+b^2+c^2+6 \geqslant 12$, or $a^2+b^2+c^2 \geqslant 6$, which was proved.

2. Seniors *March 15*

5. All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

E. Bakaev

Solution. Let's we fit this polygon into the circle. Consider each of the 50 diameters formed by the vertices of the polygon. Note that if there are two diameters among them for which the vertex color set matches, then the four ends of these diameters form a rectangle that matches the condition. There are $10 \cdot 9 : 2 = 45$ options to paint the ends of the diameter in two different colors (without taking into account the order). This means that if a suitable rectangle has not yet been found, there will be at least five diameters whose ends are of the same color. Choose any two, and their ends form a suitable rectangle.

6. Morteza wishes to take two real numbers S and P, and then to arrange six pairwise distinct real numbers on a circle so that for each three consecutive numbers at least one of the two following conditions holds:

1) their sum equals S;

2) their product equals P.

Determine if Morteza's wish could be fulfilled.

M. Saghafian

Answer. No, Morteza's wish can not be fulfilled.

Solution. Suppose one can select numbers $a_1, \ldots, a_6$. Call triple of cosequent numbers *red* if their sum equals S and *blue* in another case, this means that the product in this triple equals P.

1) Two triples with two common numbers can not be both red. Assume the opposite, $a_1+a_2+a_3 = a_2+a_3+a_4 = S$, then $a_1 = a_4$ and we get a contradiction.

2) If two consequent triples are blue, then $P = 0$, because if $a_1a_2a_3 = a_2a_3a_4 = P \neq 0$, then $a_1 = a_4$.

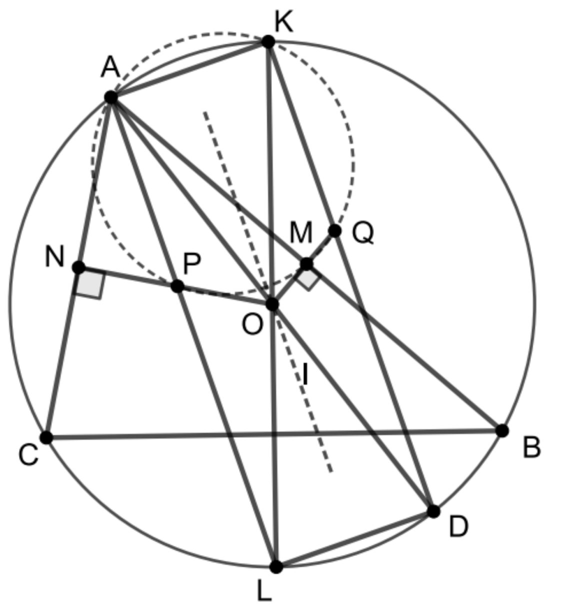
3) Case $P = 0$ is impossible. Assume the opposite, then there is at most one zero among a_i, that's why there are four consequent nonzero numbers. The two triples among these four numbers are both red, but this contradicts 1)

4) From 1), 2), 3) we get that the colours of (a_1, a_2, a_3), (a_2, a_3, a_4), (a_3, a_4, a_5), $(a_4, , a_5, a_6)$, (a_5, a_6, a_1), (a_6, a_1, a_2) alternate.

5) Without loss of generality assert that (a_1, a_2, a_3), (a_3, a_4, a_5), (a_5, a_6, a_1) are red triples and other triples are blue. Then $a_1 + a_2 + a_3 = a_3 + a_4 + a_5$, $a_1 + a_2 = a_4 + a_5$. Moreover, $a_6a_1a_2 = a_4a_5a_6 = P \neq 0$ (from 3)), that's why $a_1a_2 = a_4a_5$. Hence pairs (a_1, a_2) and (a_4, a_5) have equal sums and products. Then these pairs are the roots of the same quadratic polynomial so they are equal. This contradicts the condition.

7. In a triangle ABC with $AB \neq AC$ let M be the midpoint of AB, let K be the midpoint of the arc BAC in the circumcircle of ABC, and let the perpendicular bisector of AC meet the bisector of the angle BAC at P. Prove that A, M, K, P are concyclic.

P. Kozhevnikov



Solution. Let KL and AD be diameters of the circle (ABC). Thus $AKDL$ is a rectangle centered at the circumcenter O. By ℓ denote one of the axis of symmetry of $AKDL$ which is parallel to AL.

Note that AL is the bisector of the angle BAC. Hence $P = AL \cap ON$, where$N$ is the midpoint of AC.

The perpendicular bisectors OM and ON of AB and AC form equal angles with AL, hence OM and ON are symmetric in ℓ. This means that P and $Q = KD \cap OM$ are symmetric in ℓ, hence $AKQP$ is a rectangle, and points A, K, Q, P lie in the circle ω with diameter AQ. Since $AM \perp QM$ $(AB \perp OM)$, M also lies in ω.

Remark. The statement similar to the statement of the problem also holds: A, M, L and $ON \cap AK$ are concyclic.

8. Peter wrote 100 distinct integers on a board. Basil needs to fill the cells of a table 100×100 with integers so that the sum in each rectangle 1×3 (either vertical, or horizontal) is equal to one of the numbers written on the board. Find the greatest n such that, regardless of numbers written by Peter, Basil can fill the table so that it would contain each of numbers $1, 2, \ldots, n$ at least once (and possibly some other integers).

E. Bakaev

Answer. 6. **Solution.** Suppose that Peter wrote out the numbers a_1, a_2, ..., a_{100}. An example can be obtained by repeating the table 3×3 located below. The sum of the numbers in every rectangle 1×3 is a_1.

1	5	$a_1 - 6$
6	$a_1 - 8$	2
$a_1 - 7$	3	4

Suppose all Peter's numbers are divisible by $k > 18$. We will consider an arbitrary table filled with numbers satisfying the condition, and will show that it can not contain all the numbers from 1 to 7 simultaneously.

Let us consider an arbitrary rectangle 1×4; suppose it is filled with the numbers a, b, c, d. The sums $a+b+c$ and $b+c+d$ should be equal to some of the Peter's numbers, thus they both are divisible by k. Then their difference $a - d$ is also divisible by k. So, the numbers a and d written in the first and the last cells of this rectangle are congruent modulo k.

Let us paint the table in 9 colors in the following order:

c_1	c_2	c_3	c_1	c_2	c_3	...
c_4	c_5	c_6	c_4	c_5	c_6	...
c_7	c_8	c_9	c_7	c_8	c_9	...
c_1	c_2	c_3	c_1	c_2	c_3	...
c_4	c_5	c_6	c_4	c_5	c_6	...
c_7	c_8	c_9	c_7	c_8	c_9	...
...	...	...	...	...	...	...

Consider an arbitrary color c_i. The two numbers in any two "adjacent"cells of this color are congruent modulo k, because they occupy the first cell and the last cell of some rectangle 1×4. Between any two cells of color c_i one can move making every step in one of the "adjacent"cells of the same color, consequently all the numbers of color c_i are congruent modulo k.

So, each number in the table is congruent modulo k to one of the numbers in the top left square 3×3. We will show that it is impossible to fill it with numbers in such a way that it contains every remainder from 1 to 7 modulo k. Let us assume the opposite. Suppose that numbers with remainders from 1 to 7 occupy some 7 cells of the square 3×3. Then some 3 of them form a rectangle 1×3 and their sum should be divisible by k. But sum of this three remainders is greater than 0 and not greater than $5+6+7 = 18$, hence less than k. So it is not divisible by k; contradiction.