

## Juniors. First day

### March 14

1. Using one magic nut, Wicked Witch can either turn a flea into a beetle or a spider into a bug; but using one magic acorn, she can either turn a flea into a spider or a beetle into a bug. In the evening Wicked Witch had spent 20 magic nuts and 23 magic acorns. By a sequence of these actions, the number of beetles increased by 5. Determine what was the change in the number of spiders.

2. Let  $\omega_1$  and  $\omega_2$  be two non-intersecting circles. Let one of its internal tangents touch  $\omega_1$  and  $\omega_2$  at  $A_1$  and  $A_2$ , respectively, and let one of its external tangents touch  $\omega_1$  and  $\omega_2$  at  $B_1$  and  $B_2$ , respectively. Prove that if  $A_1B_2 = A_2B_1$ , then  $A_1B_2 \perp A_2B_1$ .

3. Let  $a_n$  be a sequence given by  $a_1 = 18$ , and  $a_n = a_{n-1}^2 + 6a_{n-1}$ , for  $n > 1$ . Prove that this sequence contains no perfect powers.

4. Positive integers  $n, k > 1$  are given. Pasha and Vova play a game on a board  $n \times k$ . Pasha begins, and thereafter they alternate using the following moves. On each move a player should mark a border of length 1 between two adjacent cells. The player loses if, after his move, it is not possible to start from the bottom left cell and reach the top right cell moving through edges of adjacent cells without crossing any marked border. Determine which of the players has a winning strategy.

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### March 15

5. Find the number of pairs of positive integers  $a$  and  $b$  such that  $a \leq 100\,000$ ,  $b \leq 100\,000$ , and

$$\frac{a^3 - b}{a^3 + b} = \frac{b^2 - a^2}{b^2 + a^2}.$$

6. All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

7. An equilateral triangle  $ABC$  is given. Points  $K$  and  $N$  lie in the segment  $AB$ , a point  $L$  lies in the segment  $AC$ , and a point  $M$  lies in the segment  $BC$  so that  $CL = AK$ ,  $CM = BN$ ,  $ML = KN$ . Prove that  $KL \parallel MN$ .

8. Let the real numbers  $a$ ,  $b$ , and  $c$  satisfy

$$abc + a + b + c = ab + bc + ca + 5.$$

Find the minimal possible value of  $a^2 + b^2 + c^2$ .

## Seniors. First day

### March 14

1. Determine if there exists a finite set  $A$  of positive integers satisfying the following condition: for each  $a \in A$  at least one of two numbers  $2a$  and  $a/3$  belongs to  $A$ .

2. Let  $\omega_1$  and  $\omega_2$  be two non-intersecting circles. Let one of its internal tangents touch  $\omega_1$  and  $\omega_2$  at  $A_1$  and  $A_2$ , respectively, and let one of its external tangents touch  $\omega_1$  and  $\omega_2$  at  $B_1$  and  $B_2$ , respectively. Prove that if  $A_1B_2 \perp A_2B_1$ , then  $A_1B_2 = A_2B_1$ .

3. Peter and Basil play the following game on a horizontal table  $1 \times 2019$ . Initially Peter chooses  $n$  positive integers and writes them on a board. After that Basil puts a coin in one of the cells. Then at each move, Peter announces a number  $s$  among the numbers written on the board, and Basil needs to rearrange the coin by  $s$  cells, if it is possible: either to the left, or to the right, by his decision. In case it is not possible to rearrange the coin by  $s$  cells neither to the left, nor to the right, the coin stays in the current cell. Find the least  $n$  such that Peter can play so that the coin will visit all the cells, regardless of the way Basil plays.

4. Find all functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that for all positive integers  $m$  and  $n$  the number  $f(m) + n - m$  is divisible by  $f(n)$ .

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5. All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

6. Morteza wishes to take two real numbers  $S$  and  $P$ , and then to arrange six pairwise distinct real numbers on a circle so that for each three consecutive numbers at least one of the two following conditions holds:

- 1) their sum equals  $S$ ;
- 2) their product equals  $P$ .

Determine if Morteza's wish could be fulfilled.

7. In a triangle  $ABC$  with  $AB \neq AC$  let  $M$  be the midpoint of  $AB$ , let  $K$  be the midpoint of the arc  $BAC$  in the circumcircle of  $ABC$ , and let the perpendicular bisector of  $AC$  meet the bisector of the angle  $BAC$  at  $P$ . Prove that  $A, M, K, P$  are concyclic.

8. Peter wrote 100 distinct integers on a board. Basil needs to fill the cells of a table  $100 \times 100$  with integers so that the sum in each rectangle  $1 \times 3$  (either vertical, or horizontal) is equal to one of the numbers written on the board. Find the greatest  $n$  such that, regardless of numbers written by Peter, Basil can fill the table so that it would contain each of numbers  $1, 2, \dots, n$  at least once (and possibly some other integers).