

VI Caucasus Mathematic Olympiad
Maykop, March 12–17, 2021 year



Juniors. Day 1
March 13

1. Let a, b, c be real numbers such that $a^2 + b = c^2$, $b^2 + c = a^2$, $c^2 + a = b^2$. Find all possible values of abc .
2. In a triangle ABC let K be a point on the median BM such that $CK = CM$. It appears that $\angle CBM = 2\angle ABM$. Prove that $BC = MK$.
3. We have $n > 2$ non-zero integers such that each one of them is divisible by the sum of the other $n - 1$ numbers. Prove that the sum of all the given numbers is zero.
4. A square grid $2n \times 2n$ is constructed of matches (each match is a segment of length 1). By one move Peter can choose a vertex which (at this moment) is the endpoint of 3 or 4 matches and delete two matches whose union is a segment of length 2. Find the least possible number of matches that could remain after a number of Peter's moves.

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Seniors. Day 1
March 13

1. Integers from 1 to 100 are placed in a row in some order. Let us call a number *large-right*, if it is greater than each number to the right of it; let us call a number *large-left*, if it is greater than each number to the left of it. It appears that in the row there are exactly k large-right numbers and exactly k large-left numbers. Find the maximal possible value of k .

2. We have $n > 2$ non-zero integers such that each one of them is divisible by the sum of the other $n - 1$ numbers. Prove that the sum of all the given numbers is zero.

3. Let $n \geq 3$ be a positive integer. In the plane n points which are not all collinear are marked. Find the least possible number of triangles whose vertices are all marked.

(Recall that the vertices of a triangle are not collinear.)

4. In an acute triangle ABC let AH_a and BH_b be altitudes. Let H_aH_b intersect the circumcircle of ABC at P and Q . Let A' be the reflection of A in BC , and let B' be the reflection of B in CA . Prove that A', B', P, Q are concyclic.

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Juniors. Day 2
March 14

5. Let a, b, c be positive integers such that the product

$$\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a)$$

is a perfect square. Prove that the product

$$\operatorname{lcm}(a, b) \cdot \operatorname{lcm}(b, c) \cdot \operatorname{lcm}(c, a)$$

is also a perfect square.

6. A row of 2021 balls is given. Pasha and Vova play a game, taking turns to perform moves; Pasha begins. On each turn a boy should paint a non-painted ball in one of the three available colors: red, yellow, or green (initially all balls are non-painted). When all the balls are colored, Pasha wins, if there are three consecutive balls of different colors; otherwise Vova wins. Who has a winning strategy?

7. An acute triangle ABC is given. Let AD be its altitude, let H and O be its orthocenter and its circumcenter, respectively. Let K be the point on the segment AH with $AK = HD$; let L be the point on the segment CD with $CL = DB$. Prove that line KL passes through O .

8. Let us call a set of positive integers *nice*, if its number of elements is equal to the average of all its elements. Call a number n *amazing*, if one can partition the set $\{1, 2, \dots, n\}$ into nice subsets.

a) Prove that any perfect square is amazing.

b) Prove that there exist infinitely many positive integers which are not amazing.

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Seniors. Day 2
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5. A triangle Δ with sidelengths $a \leq b \leq c$ is given. It appears that it is impossible to construct a triangle from three segments whose lengths are equal to the altitudes of Δ . Prove that $b^2 > ac$.

6. A row of 2021 balls is given. Pasha and Vova play a game, taking turns to perform moves; Pasha begins. On each turn a boy should paint a non-painted ball in one of the three available colors: red, yellow, or green (initially all balls are non-painted). When all the balls are colored, Pasha wins, if there are three consecutive balls of different colors; otherwise Vova wins. Who has a winning strategy?

7. 4 tokens are placed in the plane. If the tokens are now at the vertices of a convex quadrilateral P , then the following move could be performed: choose one of the tokens and shift it in the direction perpendicular to the diagonal of P not containing this token; while shifting tokens it is prohibited to get three collinear tokens.

Suppose that initially tokens were at the vertices of a rectangle Π , and after a number of moves tokens were at the vertices of one another rectangle Π' such that Π' is similar to Π but not equal to Π . Prove that Π is a square.

8. An infinite table whose rows and columns are numbered with positive integers, is given. For a sequence of functions $f_1(x), f_2(x), \dots$ let us place the number $f_i(j)$ into the cell (i, j) of the table (for all $i, j \in \mathbb{N}$). A sequence $f_1(x), f_2(x), \dots$ is said to be *nice*, if all the numbers in the table are positive integers, and each positive integer appears exactly once. Determine if there exists a nice sequence of functions $f_1(x), f_2(x), \dots$, such that each $f_i(x)$ is a polynomial of degree 101 with integer coefficients and its leading coefficient equals to 1.