VI Caucasus Mathematic Olympiad

Solutions
Day 2

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

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1. Юниоры *14 марта*

5. Let a, b, c be positive integers such that the product

$$gcd(a, b) \cdot gcd(b, c) \cdot gcd(c, a)$$

is a perfect square. Prove that the product

$$lcm(a, b) \cdot lcm(b, c) \cdot lcm(c, a)$$

is also a perfect square.

Saghafian M., Kozhevnikiov P.

Solution 1. For each pair of these numbers, we use the well-known equality $gcd(a, b) \cdot lcm(a, b) = ab$. We get that the product of two products from the problem condition is $(ab)(bc)(ca) = (abc)^2$. Since it is given that one of the products is — an exact square, then the other is — too.

Solution 2. Note that a number is an exact square if and only if every prime number is included in its expansion to an even degree. Let an arbitrary prime number p be included in the expansion of the numbers a, b, c in the powers of α, β, γ, respectively, WLOG, we assume that $\alpha \leqslant \beta \leqslant \gamma$. Then p is included in the decomposition of numbers $lcm(a, b)$, $lcm(b, c)$, $lcm(c, a)$, respectively, in degrees β, γ, γ, and in the product of these lcm — in degree of $\beta + 2\gamma$. Similarly, p is included in the decomposition of numbers $gcd(a, b)$, $gcd(b, c)$, $gcd(c, a)$, respectively, in degrees α, β, α, and in the product of these gcd — in degree of $\beta + 2\alpha$. By the condition $\beta + 2\alpha$ is even, from where β is even, and so $\beta + 2\gamma$ is even. Having carried out the same reasoning for all p, we get the required value.

6. A row of 2021 balls is given. Pasha and Vova play a game, taking turns to perform moves; Pasha begins. On each turn a boy should paint a non-painted ball in one of the three available colors: red, yellow, or green (initially all balls are non-painted). When all the balls are colored, Pasha wins, if there are three consecutive balls of different colors; otherwise Vova wins. Who has a winning strategy?

S. Luchinin

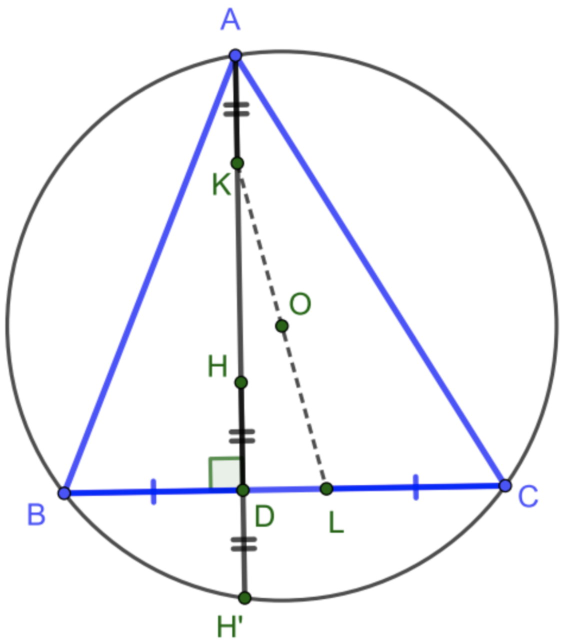
Answer. Pasha. **Solution.** Here is one of the possible winning strategies for Pasha. Let's number the balls in a row with numbers from 1 to 2021. The first move is to paint red the ball number 1011 (the middle one in the whole row). Let Vova, WLOG, make his move in the left half. Then, with the second move, Pasha paints the ball with the number 1014 in blue.

So Pasha, after his first two moves, got the situation to R ◯ ◯ B with. If Vova paints one of the two balls between the painted Pasha in red or blue, then Pasha will be able to immediately paint the remaining ball in green so that three consecutive multi-colored balls are formed. If Vova paints one of the two balls green, then Pasha will paint the remaining ball red in response, and again a multi-colored three balls lying in a row will be formed.

It remains to note that Pasha can force Vova to make a move between the colored balls of the first two moves. Pasha himself will not go there, and after painting all the other balls according to parity, Vova will move. So Pasha wins.

7. An acute triangle ABC is given. Let AD be its altitude, let H and O be its orthocenter and its circumcenter, respectively. Let K be the point on the segment AH with $AK = HD$; let L be the point on the segment CD with $CL = DB$. Prove that line KL passes through O.

Bakaev E.



Solution. Let us show that O is the midpoint of the hypothenuse KL in rectangular triangle KDL. It is sufficient to show that O lies on the perpendicular bisectors of DL and DK.

The perpendicular bisector to DL coincides to the perpendicular bisector of BC, hence O lies on it.

Further, let H' be the reflection of H in BC. It is known that H' lies on the circle (ABC). And, since $H'D = HD = AK$, the perpendicular bisector of DK coincides with the perpendicular bisector to AH', hence O belongs to it.

Remark. The fact that O lies on the perpendicular bisector of DK, could be proved in many ways, e.g., using the known fact that the distance from O to BC equals to $AH/2$.

8. Let us call a set of positive integers *nice*, if its number of elements is equal to the average of all its elements. Call a number n *amazing*, if one can partition the set $\{1, 2, \ldots, n\}$ into nice subsets.

a) Prove that any perfect square is amazing.

b) Prove that there exist infinitely many positive integers which are not amazing.

Saghafian M.

Solution. (a) Let's prove that the number $n = k^2$ is amazing. If $k = 1$, then you do not even need to split it, the set itself is immediately nice. For the remaining k, we prove that there exists a partition of the set $\{1, 2, \ldots k^2\}$ into two nice subsets. First, we type a subset of the size $\dfrac{k(k-1)}{2}$ and with the same arithmetic mean.

If $\dfrac{k(k-1)}{2}$ is odd and is equal to $2t+1$, then you can take numbers from $t+1$ to $3t+1$. It is easy to see that there, in addition to the number $2t+1$, t pairs of numbers with a half-sum equal to $2t+1$. It is also easy to see that $3t+1 \leqslant k^2$, since $t = \dfrac{k^2-k-2}{4}$ and $3t+1 = \dfrac{3k^2-3k-2}{4}$.

If $\dfrac{k(k-1)}{2}$ is even and equals $2t$, then you can take numbers from t to $2t-1$ and from $2t+1$ to $3t$. It is easy to see that here are t pairs, the average in each of which is $2t$, in addition, it is obvious that $3t < k^2$.

So, we chose the set of $\dfrac{k(k-1)}{2}$ elements and the sum of $\dfrac{k^2(k-1)^2}{4}$. There is $\dfrac{k(k+1)}{2}$ unselected numbers and their sum

$$\frac{k^2(k^2+1)}{2} - \frac{k^2(k-1)^2}{4} = \frac{2k^4+2k^2-k^4+2k^3-k^2}{4} = \frac{k^2(k+1)^2}{4}.$$

That is, there is also a nice subset left.

(b) We prove that numbers of the form $4k+2$ are not amazing. Suppose that $n = 4k+2$ is an amazing number, that is, the numbers $\{1, 2, \ldots 4k+2\}$ were able to be split into beautiful sets. Let's denote the sizes of these sets $a_1, a_2, \ldots a_t$. Then the sums of the numbers in these sets are $a_1^2, a_2^2, \ldots a_t^2$. We get two conditions: $a_1 + \ldots + a_t = 4k+2$, and $a_1^2 + \ldots + a_t^2 = 1+2+\ldots+4k+2 = (2k+1)(4k+3)$. It turns out that the sum of the numbers $a_1, \ldots, a_t$ and the sum of their squares are of different parity. But this can't be the case, since a_i^2 is the same parity as a_i.

Remark 1. The cardinality of the sets in a) could be found by solving the system $a_1 + a_2 = k^2$ and $a_1^2 + a_2^2 = \dfrac{k^2(k^2+1)}{2}$.

Remark 2. The first beautiful set in a) could have been explicitly omitted. It is easy to understand that the sum of$\frac{k^2-k}{2}$ numbers from the set $\{1, \ldots k^2\}$ can take any intermediate value.

2. Сеньоры *14 марта*

5. A triangle Δ with sidelengths $a \leqslant b \leqslant c$ is given. It appears that it is impossible to construct a triangle from three segments whose lengths are equal to the altitudes of Δ. Prove that $b^2 > ac$.

Kozhevnikov P.

Solution. Let S be the area of the triangle, hence its altitudes are equal to $h_a = \frac{2S}{a}$, $h_b = \frac{2S}{b}$, $h_c = \frac{2S}{c}$, we have $h_a \geqslant h_b \geqslant h_c$.

By conditions of the problem, $h_a > h_b + h_c$, hence

$$\frac{1}{a} \geqslant \frac{1}{b} + \frac{1}{c}. \qquad (1)$$

Multiplying (1) by inequality $a + b > c$, we have $1 + \frac{b}{a} > \frac{c}{b} + 1$, hence $\frac{b}{a} > \frac{c}{b}$ and $b^2 > ac$.

Remark. From (1) one can derive required in many ways. E.g., from (1) it follows that $\frac{1}{a} \geqslant \frac{1}{b} + \frac{1}{a+b}$, which is equivalent to $b^2 > a(a + b)$, and then, by $a + b > c$, the required $b^2 > ac$ follows.

6. A row of 2021 balls is given. Pasha and Vova play a game, taking turns to perform moves; Pasha begins. On each turn a boy should paint a non-painted ball in one of the three available colors: red, yellow, or green (initially all balls are non-painted). When all the balls are colored, Pasha wins, if there are three consecutive balls of different colors; otherwise Vova wins. Who has a winning strategy?

S. Luchinin

Answer. Pasha. **Solution.** Here is one of the possible winning strategies for Pasha. Let's number the balls in a row with numbers from 1 to 2021. The first move is to paint red the ball number 1011 (the middle one in the whole row). Let Vova, WLOG, make his move in the left half. Then, with the second move, Pasha paints the ball with the number 1014 in blue.

So Pasha, after his first two moves, got the situation to R ◯ ◯ B with. If Vova paints one of the two balls between the painted Pasha in red or blue, then Pasha will be able to immediately paint the remaining ball in green so that three consecutive multi-colored balls are formed. If Vova paints one of the two balls green, then Pasha will paint the remaining ball red in response, and again a multi-colored three balls lying in a row will be formed.

It remains to note that Pasha can force Vova to make a move between the colored balls of the first two moves. Pasha himself will not go there, and after painting all the other balls according to parity, Vova will move. So Pasha wins.

7. 4 tokens are placed in the plane. If the tokens are now at the vertices of a convex quadrilateral P, then the following move could be performed: choose one of the tokens and shift it in the direction perpendicular to the diagonal of P not containing this token; while shifting tokens it is prohibited to get three collinear tokens.

Suppose that initially tokens were at the vertices of a rectangle Π, and after a number of moves tokens were at the vertices of one another rectangle Π' such that Π' is similar to Π but not equal to Π. Prove that Π is a square.

Bakaev E.

Solution. Let $ABCD$ be the quadrilateral with its vertices at tokens. If we shift, say, A along the perpendicular to BD, then $AB^2 - DA^2$ is invariant. Hence, under any operation $f(ABCD) = AB^2 - BC^2 + CD^2 - DA^2$ is invariant.

If $A'B'C'D'$ is similar to $ABCD$ with ratio k, $f(A'B'C'D') = \pm k^2 \cdot f(ABCD)$. Therefore, under conditions of the problem, we have $f(\Pi) = 0$. To complete the solution, it suffices that for Π which is not a square, we have $f(\Pi) \neq 0$.

Remark. Note that it is well known that $AB^2 - BC^2 + CD^2 - DA^2 = 0$ is equivalent to $AC \perp BD$.

Instead of invariant f one can use some other versions of invariant, like $\overrightarrow{AC} \cdot \overrightarrow{BD}$.

8. An infinite table whose rows and columns are numbered with positive integers, is given. For a sequence of functions $f_1(x), f_2(x), \ldots$ let us place the number $f_i(j)$ into the cell (i, j) of the table (for all $i, j \in \mathbb{N}$). A sequence $f_1(x), f_2(x), \ldots$ is said to be *nice*, if all the numbers in the table are positive integers, and each positive integer appears exactly once. Determine if there exists a nice sequence of functions $f_1(x), f_2(x), \ldots$, such that each $f_i(x)$ is a polynomial of degree 101 with integer coefficients and its leading coefficient equals to 1.

Kozhevnikov P.

Answer. Yes, exists.

Solution. let us call an increasing sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ *realizable*, if $a_i = f(i)$ for all $i = 1, 2, \ldots$ for some (realizing) polynomial $f(x)$ of degree 101 with integer coefficients and its leading coefficient 1. Let us show that $\mathbb{N}$ can be represented as a union of (infinitely many) pairwise disjoint sequences; from that the positive answer to the question of the problem follows.

Note that a shift of a realizable sequence $a_1, a_2, a_3, \ldots$ (i. e. the sequence $a_1 + c$, $a_2 + c$, $a_3 + c$, $\ldots$ for some integer c) is also a realizable sequence (which realized by the polynomial $f(x) + c$, where f realizes $a_1, a_2, \ldots$). A «tail» $a_k, a_{k+1}, a_{k+2}, \ldots$ of a realizable sequence $a_1, a_2, a_3, \ldots$ is a realizable sequence: it is realized by the polynomial $f(x + k - 1)$ (which obviously has degree $deg f(x) = 101$, integer coefficients and the leading coefficient 1).

Lemma. Assume that for an increasing sequence of integers $A = (a_1, a_2, a_3, \ldots)$ the condition $0 < a_2 - a_1 < a_3 - a_2 < a_4 - a_3 < \ldots$ holds. Then $\mathbb{N}$ could be partitioned into disjoint sequnces, each is a tail of a shift of A.

Proof. Replacing A by its shift, we can set $a_1 = 0$. Denote by A_c the shift of A by c, i. e. the sequence $a_1 + c, a_2 + c, a_3 + c, \ldots$. From each A_c erase each term $a_i + c > a_{i+1}$. After erasing we result with $a_i + c$, for which $a_{i+1} - a_i \geqslant c$. This is a tail of A_c, by $0 < a_2 - a_1 < a_3 - a_2 < a_4 - a_3 < \ldots$, and the union of these tails is the required partition of $\mathbb{N}$. Indeed, let us consider positive integers from $a_i + 1, a_2 + 2, \ldots, a_{i+1}$: by the construction, each of them belongs to exactly one of tails defined above. Lemma is proved.

By lemma, now it suffices to present at least one realized sequence sayisfying $0 < a_2 - a_1 < a_3 - a_2 < a_4 - a_3 < \ldots$. Obviously, $f(1)$, $f(2)$, $f(3)$, $\ldots$, where $f(x) = x^{101}$, satisfies the condition.