VII Caucasus Mathematic Olympiad

Solutions
Day 1

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Maykop
Adygea

1. Juniors *March 12*

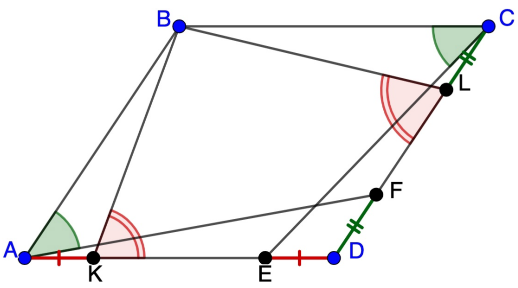
1. Positive integers a, b, c are given. It is known that $\frac{c}{b} = \frac{b}{a}$, and the number $b^2 - a - c + 1$ is a prime. Prove that a and c are doubled squares of positive integers.

Antropov A., Belov D.

Solution. Let us rewrite the condition $\frac{c}{b} = \frac{b}{a}$ as $b^2 = ac$. Substituting that in the second condition, we obtain that $ac - a - c + 1 = (a-1)(c-1)$ is a prime. Since a and c are positive integers, this is possible if either $a-1$ or $c-1$ equals 1. Without loss of generality, assume that $a-1=1$. Hence, $a = 2$ and $b^2 = ac = 2c$. Thus, $b = 2k$ for some positive integer k and $c = 2k^2$. At last, we have that $a = 2$ and $c = 2k^2$ are doubled squares of positive integers, we are done.

2. In parallelogram $ABCD$, points E and F on segments AD and CD are such that $\angle BCE = \angle BAF$. Points K and L on segments AD and CD are such that $AK = ED$ and $CL = FD$. Prove that $\angle BKD = \angle BLD$.

Kukharchuk I.



Solution. Since $AB \parallel CD$ and $BC \parallel AD$, we have $\angle AFD = \angle BAF = \angle BCE = \angle CED$. Thus, the triangles AFD and CED are similar as they share the angle D. Using the equalities $AK = ED$ and $CL = FD$, we obtain $\frac{AK}{CL} = \frac{ED}{FD} = \frac{DC}{AD}$.

Since $ABCD$ is a parallelogram, we have the equalities $AB = CD$, $BC = AD$, and $\angle BAK = \angle BCL$. Thus $\frac{AK}{CL} = \frac{DC}{AD} = \frac{AB}{BC}$, and so, the triangles BAK and BCL are similar. This yields that $\angle BKA = \angle BLC$, and so, $\angle BKD = \angle BLD$, we are done.

3. Pete wrote down 21 pairwise distinct positive integers, each not greater than $1,000,000$. For every pair (a, b) of numbers written down by Pete, Nick wrote the number

$$F(a; b) = a + b - \gcd(a; b)$$

on his piece of paper. Prove that one of Nick's numbers differs from all of Pete's numbers.

Agakhanov N.

Solution. Denote the numbers written by Pete by

$$a_1 < a_2 < \ldots < a_{21}.$$

Suppose the contrary, that is, every number written by Nick equals to some number written by Pete. The inequality $\gcd(a;b) \leqslant \min(a;b)$ implies that

$$F(a_{20}, a_{21}) = a_{20} + a_{21} - \gcd(a_{20}; a_{21}) \geqslant a_{20} + a_{21} - a_{20} = a_{21},$$

Hence the number $F(a_{20}; a_{21})$ written by Nick must be equal to a_{21}. Therefore, $\gcd(a_{20}; a_{21}) = a_{20}$ and a_{21} is divisible by a_{20}.

Let us show that $a_{\ell+1}$ is divisible by a_ℓ for all $1 \leqslant \ell \leqslant 19$. Suppose the contrary and k is the largest index that this divisibility does not hold. Since a_{k+1} is not divisible by a_k, then $\gcd(a_k; a_{k+1}) < a_k$, and so $F(a_k; a_{k+1}) > a_{k+1}$. On the other hand, we have

$$F(a_k; a_{k+1}) = a_k + a_{k+1} - \gcd(a_k; a_{k+1}) < a_k + a_{k+1} < 2a_{k+1} \leqslant a_{k+2}.$$

Here the last inequality holds as a_{k+2} is divisible by a_{k+1} and a_{k+2} is larger than a_{k+1}. Thus,

$$a_{k+1} < F(a_k; a_{k+1}) < a_{k+2},$$

a contradiction with the fact that the $F(a_k; a_{k+1})$ does not occur among numbers written by Pete.

Therefore, we proved that each next number written by Pete is divisible by the previous one. Since all numbers a_k are different, we get that $a_{k+1} \geqslant a_k$, and hence, $a_{21} \geqslant 2^{20} > 1\,000\,000$, a contradiction.

4. Do there exist 2021 points with integer coordinates on the plane such that the pairwise distances between them are pairwise distinct consecutive integers?

Saghafian M.

Answer. No, does not exist.

Solution. Suppose the contrary. There are $\frac{2021 \cdot 2020}{2} = 2\,041\,210 = n$ pairwise distances equal to consecutive positive integers. Since among these numbers there are $1\,020\,605$ odd numbers, the total sum of all distances is odd as well.

On the other hand, denote by x the number of points with odd sum of coordinates. Analogously, denote by $2021 - x$ the number of points with even sum of coordinates. Let us show that the distances between points is odd if and only if their sums of coordinates are of different parity. Consider two points (x_1, y_1) and (x_2, y_2) that are at odd distance apart. If the distance between them is odd, then one of the numbers $x_1 - x_2$ or $y_1 - y_2$ is odd and another is even. Therefore, we obtain $x_1 + y_1 - (x_2 + y_2)$ is odd. This means that the sums of the coordinates of the points (x_1, y_1) and (x_2, y_2) have different parity.

Thus, there occur exactly $x \cdot (2021 - x)$ odd distances. Since $x \cdot (2021 - x)$ is even for every x, the sum of all distances is even as well, a contradiction.

2. Seniors *March 12*

1. Given a rectangular table with 2 rows and 100 columns. Dima fills the cells of the first row with numbers 1, 2 or 3. Prove that Alex can fill the cells of the second row with numbers 1, 2, 3 in such a way that the numbers in each column are different and the sum of the numbers in the second row equals 200.

Saghafian M.

Solution. If in the first row there are $2k$ cells with 2, then Alex writes 1 under k of them and 3 under other k of them. Also, he writes 2 under every other cell of the first row. Therefore, the sum of all numbers of the second row equals 200.

If in the first row there are $2k+1$ cells with 2, then there is at least one cell with 1 or 3 in the first row as there are 200 (even) cells in the row. Without loss of generality, we can assume that there is a cell with 1. In that case Alex writes 3 under it, and then he writes 2 under the rest cells with 1 and 3. After that he writes 1 under k cells with 2 and 3 under other $k+1$ cells with 2. Thus, in the second row 1 and 3 occurs the same number of time, and so, the sum of numbers in the second row is 200.

2. Prove that infinitely many positive integer numbers can be represented as $(a-1)/b+(b-1)/c+(c-1)/a$, where a, b and c are pairwise distinct natural numbers greater than 1.

Emelyanov L., Kukharchuk I.

Solution. Given a positive integer, consider the distinct numbers $a = 12k+2$, $b = 6k+1$, $c = 3$. Then the desired expression equals $\frac{12k+1}{6k+1} + \frac{6k}{3} + \frac{2}{12k+2} = 2k+2$. Since there are infinitely many even integers larger 2 and each of them can be written in that form, we are done.

3. Do there exist 100 points on the plane such that the pairwise distances between them are pairwise distinct consecutive integer numbers larger than 2022?

Bragin V., Saghafian M.

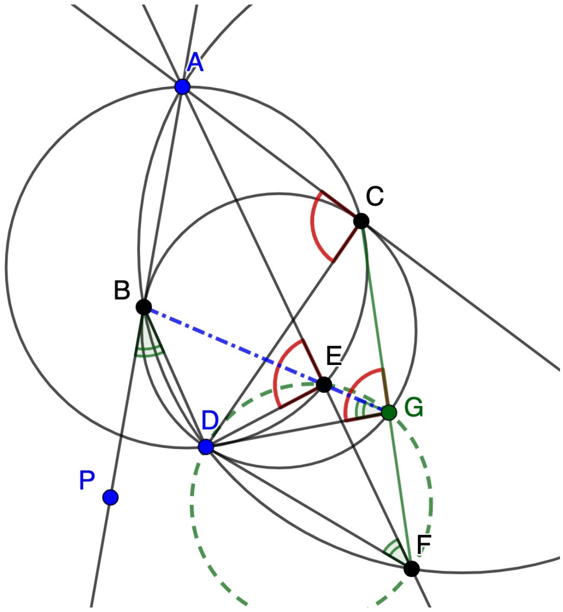
Answer. No, there are no 100 such points.

Solution. Suppose the contrary. Hence, there occurs the following distances between the points: $n+1, n+2, \ldots, n+4950$ for some positive integer $n \geqslant 2022$. If we draw the disc of radius $\frac{n}{2}$ at every point, then these discs do not overlap.

On other hand, consider two points A and B that are at the maximal distance $n+4950$. Hence, all the points lie in the disc of radius $n+4950$ with center at A. Therefore, all discs of radius $\frac{n}{2}$ lie in the disc of radius $\frac{3n}{2}+4950$ with center at A. Thus, the the sum of the area of smaller discs does not exceed the area of the big one. So, we obtain the inequality $\pi\left(\frac{3n}{2}+4950\right)^2 \geqslant 100\cdot\pi\cdot\frac{n^2}{4}$, which is equivalen to $(3n+9900)^2 \geqslant 100n^2$ and hence $9900 \geqslant 7n$, which is wrong as $n \geqslant 2022$.

4. Circle ω is tangent to the sides of an acute angle with vertex A at points B and C. Let D be an arbitrary point on the major arc BC of the circle ω. Points E and F are chosen inside the angle DAC so that the quadrilaterals $ABDF$ and $ACED$ are inscribed and the points A, E, F lie on the same straight line. Prove that the lines BE and CF intersect at ω.

Kukharchuk I.



Solution. Let the line CF intersects the circle ω at the points C and G. Since the quadrangle $ACED$ is incribed and the line AC touches ω, we obtain $\angle AED = \angle ACD = \angle CGD$. Therefore, $\angle DEF = 180^\circ - \angle AED = 180^\circ - \angle CGD = \angle DGF$, and so, the quadrilateral $DEGF$ is incribed.

Let P be any point lying on the ray AB out of the segment AB. Thus since the quadrilaterals $DEGF$ and $ABDF$ are inscribed and the line PB touches ω, we have

$$\angle EGD = \angle EFD = 180^\circ - \angle DBA = \angle PBD = \angle BGD$$

Therefore, the points B, E, and G lye on the same line. Thus the lines BE and CF intersects at the point G on ω, we are done.