VII Caucasus Mathematic Olympiad

Solutions
Day 2

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

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Maykop
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1. Juniors *March 13*

5. Let S be the set of all 5^6 positive integers, whose decimal representation consists of exactly 6 odd digits. Find the number of solutions (x, y, z) of the equation $x + y = 10z$, where $x \in S$, $y \in S$, $z \in S$.

Kozhevnikov P.

Answer. $5 \cdot 15^2 = 3796875$. **Solution.** Let us study the properties of x and y. First, the sum of their last digits must be equal to 10; this corresponds to the pairs of digits (1; 9), (3; 7), (5; 5); (7; 3), (9; 1). There are 5 possible pairs. Second, the sum of digits of x and y on k-th position ($1 \leqslant k \leqslant 5$) must be more than 10 as the k-th position of $10z$ is odd. This corresponds to the pairs of digits (1; 9), (3; 9), (3; 7), (5; 9), (5; 7), (5; 5), (7; 9), (7; 7), (7; 5), (7; 3), (9; 9), (9; 7), (9; 5), (9; 3), (9; 1). There are 15 possible pairs. Therefore, there are 5 possibilities for the last digits of x and y and there are 15 possibilities for every other pair of digits of x and y.Thus, there are $5 \cdot 15^5$ possible pairs (x, y). Clearly, every of these pairs corresponds to a suitable triple (x, y, z).

6. 16 NHL teams in the first playoff round divided in pairs and to play series until 4 wins (thus the series could finish with score 4-0, 4-1, 4-2, or 4-3). After that 8 winners of the series play the second playoff round divided into 4 pairs to play series until 4 wins, and so on. After all the final round is over, it happens that k teams have non-negative balance of wins (for example, the team that won in the first round with a score of 4-2 and lost in the second with a score of 4-3 fits the condition: it has $4 + 3 = 7$ wins and $2 + 4 = 6$ losses). Find the least possible k.

Kozhevnikov P.

Answer. 2.

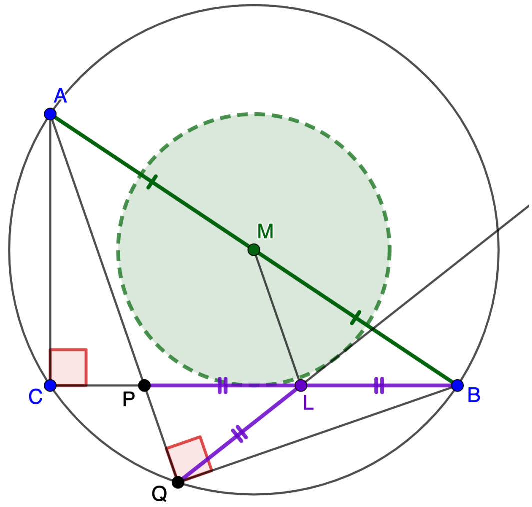
Solution. *Example.* Assume that the tournament is organized as follows. In the first round, all series ended with $4 - 3$, in the second round, all series ended with $4 - 2$, and in the third round, all ended with $4 - 0$. Then every team that left before the final has fewer wins than losses. The finalists have 7 more wins at the time of entering the final, and so, they must have no more losses after any result of the final series.

Let us show that there are at least 2 teams that have wins no less than wins. Notice that the team that wins the tournament wins all series definitely has more wins than losses. Suppose that the second-place team has more losses than wins. But this team won the first three rounds on the way to the final round, that is, before the final, the difference between wins and losses is at least 3. After the final, the difference decreased by no more than 4. The only possibility for the second-place team to have a negative difference is to win all series before the final series with $4-3$ and to lose the final series with $0-4$. Hence, the team that played with the second-place team in the semifinals had two winning series and one losing series ended with $3-4$, and so this team has more wins than losses.

7. Point P is chosen on the leg CB of right triangle ABC ($\angle ACB = 90°$). The line AP intersects the circumcircle of ABC at point Q. Let L be the midpoint of PB. Prove that QL is tangent to a fixed circle independent of the choice of point P.

Kukharchuk I.

Solution. Notice that $\angle PQB = \angle ACB = 90°$. Since L is the midpoint of the hypotenuse in the right triangle PQB, and so, $PL = LQ$. Hence $\angle LPQ = \angle LQP$.



Let M be the midpoint of AB. Since ML is a midsegment of the triangle ABP, we have $ML \parallel PQ$. Hence $\angle MLP = \angle MLN$, where N is an arbitrary point of the ray QL beyond the point L.

Thus the point M is equidistant from the sides of the angle CLN. Then the line QL touches the circle centered at M and with a radius equal to the distance from M to the line BC.

8. Paul can write polynomial $(x+1)^n$, expand and simplify it, and after that change every coefficient by its reciprocal. For example if $n = 3$ Paul gets $(x+1)^3 = x^3 + 3x^2 + 3x + 1$ and then $x^3 + \frac{1}{3}x^2 + \frac{1}{3}x + 1$. Prove that Paul can choose n for which the sum of Paul's polynomial coefficients is less than 2.022.

Saghafian M.

Solution. Notice that the value of any polynomial at point 1 equals the sum of its coefficients. The senior coefficient and the free term of the polynomial $(x+1)^n$ are equal to 1. Hence we have to show that the sum of all other coefficients of Pasha's polynomial less than 0.022. Since the coefficient at x^k of the polynomial $(x+1)^n$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the corresponding coefficient of Pasha's polynomial equal $\frac{k!(n-k)!}{n!}$. Denote this expression by a_k.

How does a_k change when k is increased by 1? It is multiplied by $k+1$ and divided by $(n-k)$, and so, for $k \leqslant \frac{n-1}{2}$ the sequence a_k decreases and for $k \geqslant \frac{n+1}{2}$ it increases.

The coefficients $a_2, \ldots, a_{n-2}$ do not exceed a_2. For $n \geqslant 1001$, we have

$$a_2 = \frac{2}{n(n-1)} = \frac{1}{n} \cdot \frac{2}{n-1} \leqslant \frac{1}{n} \cdot \frac{1}{500}.$$

Hence, we have for $n \geqslant 1001$ the following inequality holds
$$a_2 + a_3 + \ldots + a_{n-2} \leqslant (n-3) \cdot \frac{1}{n} \cdot \frac{1}{500} < \frac{1}{500}.$$
Thus, we obtain
$$a_1 + a_n + a_2 + a_3 + \ldots + a_{n-1} \leqslant \frac{2}{n} + \frac{1}{500} < \frac{2}{1000} + \frac{1}{500} < \frac{22}{1000}.$$
So $n = 1001$ satisfies the desired condition.

2. Seniors *March 13*

5. 16 NHL teams in the first playoff round divided in pairs and to play series until 4 wins (thus the series could finish with score 4-0, 4-1, 4-2, or 4-3). After that 8 winners of the series play the second playoff round divided into 4 pairs to play series until 4 wins, and so on. After all the final round is over, it happens that k teams have non-negative balance of wins (for example, the team that won in the first round with a score of 4-2 and lost in the second with a score of 4-3 fits the condition: it has $4+3=7$ wins and $2+4=6$ losses). Find the least possible k.

Kozhevnikov P.

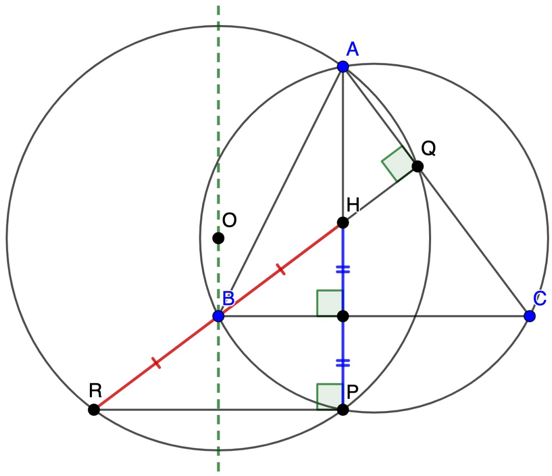
Answer. 2.

Solution. *Example.* Assume that the tournament is organized as follows. In the first round, all series ended with $4-3$, in the second round, all series ended with $4-2$, and in the third round, all ended with $4-0$. Then every team that left before the final has fewer wins than losses. The finalists have 7 more wins at the time of entering the final, and so, they must have no more losses after any result of the final series.

Let us show that there are at least 2 teams that have wins no less than wins. Notice that the team that wins the tournament wins all series definitely has more wins than losses. Suppose that the second-place team has more losses than wins. But this team won the first three rounds on the way to the final round, that is, before the final, the difference between wins and losses is at least 3. After the final, the difference decreased by no more than 4. The only possibility for the second-place team to have a negative difference is to win all series before the final series with $4-3$ and to lose the final series with $0-4$. Hence, the team that played with the second-place team in the semifinals had two winning series and one losing series ended with $3-4$, and so this team has more wins than losses.

6. Let ABC be an acute triangle. Let P be a point on the circle (ABC), and Q be a point on the segment AC such that $AP \perp BC$ and $BQ \perp AC$. Let O be the circumcenter of triangle APQ. Find the angle OBC.

Frolov I.



Answer. $\angle OBC = 90°$.

Solution. Let H be the intersection point of the altitudes of the triangle ABC. Denote by R the point centrally symmetric to the point H with respect to the point B.

Recall the following well-known fact: the point symmetric to the intersection point of the altitudes of a triangle with respect to any of its sides lies on the circumscribed circle of the triangle. Using this fact, we get that the points H and P are symmetric with respect to the line BC. Then the line BC passes through the midpoints of segments HR and HP, and so, $BC \parallel RP$. Hence $\angle APR = 90°$ and the points A, Q, P, and R lie on the same circle. Then O lies on the perpendicular bisector of the line segment RP.

On the other hand, the perpendicular bisector of the line segment RP is parallel to HP and passes through the midpoint of RP, and so, it is a midsegment of the triangle HPR. Thus, B, the midpoint of RH, lies on this perpendicular, and so $OB \perp RP$. Given $BC \parallel RP$, we obtain $OB \perp BC$ and find $\angle OBC = 90°$ as required.

7. Paul can write polynomial $(x + 1)^n$, expand and simplify it, and after that change every coefficient by its reciprocal. For example if $n = 3$ Paul gets $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$ and then $x^3 + \frac{1}{3}x^2 + \frac{1}{3}x + 1$. Prove that Paul can choose n for which the sum of Paul's polynomial coefficients is less than 2.022.

Saghafian M.

Solution. Notice that the value of any polynomial at point 1 equals the sum of its coefficients. The senior coefficient and the free term of the polynomial $(x+1)^n$ are equal to 1. Hence we have to show that the sum of all other coefficients of Pasha's polynomial less than 0.022. Since the coefficient at x^k of the polynomial $(x+1)^n$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the corresponding coefficient of Pasha's polynomial equal $\frac{k!(n-k)!}{n!}$. Denote this expression by a_k.

How does a_k change when k is increased by 1? It is multiplied by $k + 1$ and divided by $(n - k)$, and so, for $k \leqslant \frac{n-1}{2}$ the sequence a_k decreases and for $k \geqslant \frac{n+1}{2}$ it increases.

The coefficients $a_2, \ldots, a_{n-2}$ do not exceed a_2. For $n \geqslant 1001$, we have

$$a_2 = \frac{2}{n(n-1)} = \frac{1}{n} \cdot \frac{2}{n-1} \leqslant \frac{1}{n} \cdot \frac{1}{500}.$$

Hence, we have for $n \geqslant 1001$ the following inequality holds

$$a_2 + a_3 + \ldots + a_{n-2} \leqslant (n-3) \cdot \frac{1}{n} \cdot \frac{1}{500} < \frac{1}{500}.$$

Thus, we obtain

$$a_1 + a_n + a_2 + a_3 + \ldots + a_{n-1} \leqslant \frac{2}{n} + \frac{1}{500} < \frac{2}{1000} + \frac{1}{500} < \frac{22}{1000}.$$

So $n = 1001$ satisfies the desired condition.

8. There are $n > 2022$ cities in the country. Some pairs of cities are connected with straight two-ways airlines. Call the set of the cities *unlucky*, if it is impossible to color the airlines between them in two colors without monochromatic triangle (i.e. three cities A, B, C with the airlines AB, AC and BC of the same color).

The set containing all the cities is unlucky. Is there always an unlucky set containing exactly 2022 cities?

Demin D.

Answer. No, not always.

Solution. We say that the coloring in two colors of airlines in the country is proper if among any three of them there is no unlucky triangle.

Let the country consist of 2023 regions, and each region has 5 cities. Denote by R_i the ith region. Let c_i^j be the jth city of R_i. Assume that any two cities of the same region are connected with two-ways airlines. Regarding the regions R_i and R_{i+1}, we assume that there are airlines between cities with different numbers within their regions, that is, between c_i^j and c_{i+1}^k, where $j \neq k$. The airlines between R_1 and R_{2023} organized in a bit different manner: There are all airlines between all pairs of the cities exept (c_1^1, c_{2023}^1), (c_1^2, c_{2023}^2), (c_1^3, c_{2023}^4), (c_1^4, c_{2023}^5), (c_1^5, c_{2023}^3). There are no other airlines in the country.

First, notice that for any five cities pairwise connected by properly colored airlines, there are exactly two airlines of each color connecting each city with 4 others. Otherwise, we could find a city u such that there are three airlines of the same color connecting u with the cities x, y, z. Hence, the colors of airlines xy, yz, and zx must be the same, a contradiction.

Given cities $a_1, \ldots, a_5$ and $b_1, \ldots, b_5$ such that $a_1, \ldots, a_5$ are pairwise connected by airlines and $b_1, \ldots, b_5$ are pairwise connected by airlines, we say that the 5-tuple $(a_1, a_2, a_3, a_4, a_5)$ have the same coloring as the 5-tuple $(b_1, b_2, b_3, b_4, b_5)$ if for any distinct $1 \leqslant i, j \leqslant 5$ the airline $a_i a_j$ have the same color as $b_i b_j$.

Given six cities $a_1, a_2, a_3, a_4, a_5, a_6$ such that all pairs but $a_5 a_6$ are connected by an airline, consider any proper coloring of the airline. We claim that $(a_1, \ldots, a_5)$ has the same coloring as $(a_1, \ldots, a_4, a_6)$. Let us show that $a_5 a_1$ and $a_6 a_i$ are of the same color. It follows from the facts that among the airlines $a_1 a_2$, $a_1 a_3$, $a_1 a_4$, $a_1 a_5$ there are two airlines of the first color and two airlines of the second color and the same holds for the airlines a_1a_2, a_1a_3, a_1a_4, a_1a_6. Analogously, one can show that the airlines a_5a_i and a_6a_i for $1 \leqslant i \leqslant 4$ are of the same color; this finishes the proof of the claim.

Consider a proper coloring of the edges induced by $R_i \cup R_{i+1}$. One can easily find that the coloring of the following 5-tuples must be the same

$$(c_i^1, c_i^2, c_i^3, c_i^4, c_i^5), (c_{i+1}^1, c_i^2, c_i^3, c_i^4, c_i^5), (c_{i+1}^1, c_{i+1}^2, c_i^3, c_i^4, c_i^5), (c_{i+1}^1, c_{i+1}^2, c_{i+1}^3, c_i^4, c_i^5),$$

$$(c_{i+1}^1, c_{i+1}^2, c_{i+1}^3, c_{i+1}^4, c_i^5), (c_{i+1}^1, c_{i+1}^2, c_{i+1}^3, c_{i+1}^4, c_{i+1}^5).$$

Next, suppose that for the given example of cities of airlines there is a proper coloring. Since $(c_i^1, c_i^2, c_i^3, c_i^4, c_i^5)$ and $(c_{i+1}^1, c_{i+1}^2, c_{i+1}^3, c_{i+1}^4, c_{i+1}^5)$ have the same coloring for all $1 \leqslant i \leqslant 2022$, the 5-tuples

$$(c_1^1, c_1^2, c_1^3, c_1^4, c_1^5) \text{ and } (c_{2023}^1, c_{2023}^2, c_{2023}^3, c_{2023}^4, c_{2023}^5).$$

Analogously, the following 5-tuples also have the same coloring

$$(c_{2023}^1, c_{2023}^2, c_{2023}^3, c_{2023}^4, c_{2023}^5) \text{ and } (c_1^1, c_1^2, c_1^5, c_1^3, c_1^4).$$

Therefore, the color of $c_1^5c_1^3$, $c_1^3c_1^4$ и $c_1^4c_1^5$ is the same, a contradiction.

Let us show that there is no unlucky subset of 2022 cities. Clearly, for any choice 2022 cities, there is at least one region such that there is no chosen city from it. Therefore, it is enough to show that there is a proper coloring for a subset of all cities but cities of one of the regions, denoted by R_k.

For $i = 1, 2$, put $S^i := \{c_j^i \text{ for } 1 \leqslant j < k, c_j^i \text{ for } k < j \leqslant 2023\}$. If $i = 4, 5$, put $S^i := \{c_j^i \text{ for } 1 \leqslant j < k, c_j^i \text{ for } k < j \leqslant 2023\}$. If $i = 3$, put $S^i := \{c_j^i \text{ for } 1 \leqslant j < k, c_j^5 \text{ for } k < j \leqslant 2023\}$. Then, we can say that two cities from R_j and R_{j+1}, respectively, are connected by an airline if and only if they belong to different S^i.

Next, let us color in the first color the airlines between S^1 and S^2, S^2 and S^3, S^3 and S^4, S^4 and S^5, S^5 and S^1. Let us color all other airlines in the second color. Then for any three cities, we have that either two of them belong to the same S^i (and they are not connected by an airline) or they belong to different S^i. In the last case either there are two cities that are not connected by an edge or there are two colors among the corresponding airlines.