VIII Caucasus Mathematic Olympiad

Solutions
Day 1

Caucasus Mathematical Olympiad | Кавказская математическая олимпиада

Mamiy Daud
Kozhevnikov Pavel
Belov Dmitry
Bakaev Egor
Emelyanov Lev
Kukharchuk Ivan
Sukhov Kirill
Saghafian Morteza

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1. Juniors *March 10*

1. Determine the least positive integer n for which the following statement is true: the product of any n odd consecutive positive integers is divisible by 45.

Pavel Kozhevnikov

Answer. 6.

Solution. Let us give an example showing that for $n \leqslant 5$ this statement is incorrect. We take five consecutive odd numbers: 11, 13, 15, 17, 19. Only 15 is divisible by 3, but it is not divisible by 9. This means that the product of these numbers is not divisible by 9, therefore it is not divisible by $45 = 9 \cdot 5$.

Let's prove that for $n = 6$ the statement is true. It is enough to show that our product is divisible by 5 and by 9.

The first five numbers n, $n+2$, $n+4$, $n+6$, $n+8$ give different residuals modulo 5, in particular, one of the numbers is divisible by 5, which means that the product of all the numbers is divisible by 5.

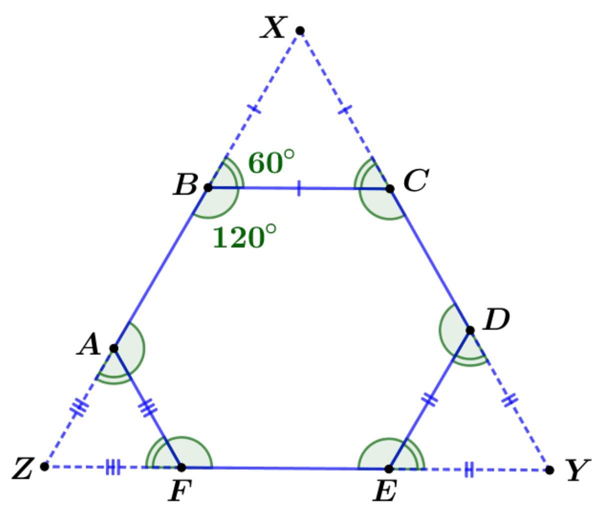
We split these 6 numbers into two triples of consecutive odd numbers so that the triple has the form n, $n+2$, $n+4$. The numbers in each group give different residuals modulo 3, in particular, one of these numbers is divisible by 3. Then the product of numbers in each triple is divisible by 3, and the product of all 6 numbers is divisible by $3 \cdot 3 = 9$.

2. In a convex hexagon the value of each angle is $120°$. The perimeter of the hexagon equals 2. Prove that this hexagon can be covered by a triangle with perimeter at most 3.

Egor Bakaev

Solution. Let us name hexagon $ABCDEF$ and without loss of generality,

$$BC + DE + AF \leqslant AB + CD + EF.$$



The perimeter of hexagon equals 2, so $BC + DE + AF \leqslant 1$.

Let's extend to the intersection pairs of sides AB and DC, CD and FE, EF and BA. Let them intersect at points X, Y and Z, respectively. The angles $\angle XBC$ and $\angle XCB$ of the triangle XBC are adjacent to the angles of the hexagon and are equal to $60°$, therefore this triangle is equilateral. Then $XB = XC = BC$, and the same is true for other triangles: YDE and ZAF. Therefore, the perimeter of the triangle XYZ is

$$AB + CD + EF + 2BC + 2DE + 2AF = 2 + BC + DE + AF \leqslant 3,$$

and this triangle covers this hexagon.

3. The numbers 1, 2, 3, ..., $2\underbrace{00\ldots0}_{100\text{ zeroes}}2$ are written on the board. Is it possible to paint half of them red and remaining ones blue, so that the sum of red numbers is divisible by the sum of blue ones?

Dmitriy Belov

Answer. No, it is not possible.

Solution. We denote the largest written number by $2n$. The minimum sum of blue numbers is

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

The maximum sum of red numbers is

$$(n+1)+(n+2)+\ldots+(n+n) = n\cdot n+1+2+\ldots+n = \frac{2n^2 + n(n+1)}{2} = \frac{3n^2+n}{2}.$$

Since $3n(n+1) > 3n^2 + n$, the ratio of the sum of red numbers to the sum of blue ones is less than three, so if the sum of red numbers is divisible by the sum of blue ones, the quotient equals 1 or 2.

In the first case, we get that the sums of the red numbers and the blue numbers must be equal, so the sum of all the numbers written on board must be even. At the same time, half of numbers, namely $1\underbrace{00\ldots0}_{100\text{ zeroes}}1$, are odd. Therefore, the sum of all numbers is actually odd, and the quotient cannot be equal to 1.

In the second case, we denote the sum of blue numbers by S. The sum of red numbers equals $2S$, and the sum of all numbers written out on board equals $3S$, so it is divisible by 3. In fact, the sum of numbers written out equals

$$1 + 2 + 3 + \ldots + 2n = \frac{2n(2n+1)}{2}.$$

The rule of divisibility by 3 states: a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. The sum of digits of number $2n = 2\underbrace{00\ldots0}_{100\text{ zeroes}}2$ equals 4, and the sum of digits of number $2n + 1$ equals 5. Therefore, both of these numbers are not divisible by 3, then the sum of all numbers written out on board is not divisible by 3, and the second case is also impossible.

4. Pasha and Vova play the game crossing out the cells of the 3×101 board by turns. At the start, the central cell is crossed out. By one move the player chooses the diagonal (there can be 1, 2 or 3 cells in the diagonal) and crosses out cells of this diagonal which are still uncrossed. At least one new cell must be crossed out by any player's move. Pasha begins, the one who can not make any move loses. Who has a winning strategy?

Igor Efremov

Answer. Vova.

Solution. Here is one of the possible Vova's winning strategies. Let's paint the cells of the board in a staggered order so that the corner cells are colored black. In odd columns, the outermost cells are colored black, and in even columns they are colored white. The central cell is in the 51st column, so it is colored white.

Each diagonal is either completely black or completely white. Both strategies will be symmetrical, but various for different colors. For white cells, we will move symmetrically relative to the central column (axial symmetry). For black cells, we will move symmetrically relative to the central cell (central symmetry).

Let's prove that Vova will always be able to make a move according to the strategy. We need to make sure that he will cross out at least one new cell every turn. For black cells, two diagonals — chosen by Pasha and centrally symmetrical to it, chosen by Vova — do not have common cells. At the same time, Pasha was able to make a move, which means that there was an uncrossed cell on the diagonal he chose. Since the situation for black cells was centrally symmetrical before Pasha's move, there was also an uncrossed cell on the Vova's diagonal before Pasha's move. Pasha could not cross it out, which means that Vova will cross out at least one new cell on his own.

For white cells, the situation is a bit different: the two diagonals passing through the central cell are symmetrical with respect to the central column, but there are no other symmetrical white diagonals on the board. Before the turn of Pasha, the situation on the white cells is axially symmetrical, that is, there are uncrossed cells on the diagonal chosen by Pasha and on the diagonal chosen according to the Vova's strategy. But the only common cell, that is axially symmetrical white diagonals can contain, is crossed out from the start. Thus, Vova will cross out at least one new cell each turn.

So, we proved that Vova always has a move according to the strategy. Since the number of moves is finite (the game will end no later than after $3 \cdot 101 = 303$ moves, when all the cells will be crossed out exactly), thus Vova wins.

2. Seniors *March 10*

1. Let n and m be positive integers, $n > m > 1$. Let n divided by m has partial quotient q and remainder r (so that $n = qm + r$, where $r \in \{0, 1, \ldots, m-1\}$; e.g., for $n = 100$, $m = 30$ we have $q = 3$, $r = 10$; for $n = 100$, $m = 25$ we have $q = 4$, $r = 0$; etc.). Let $n - 1$ divided by m has partial quotient q' and remainder r'.

a) It appears that $q + q' = r + r' = 99$. Find all possible values of n.

b) Prove that if $q + q' = r + r'$, then $2n$ is a perfect square.

Nazar Agakhanov

Answer. a) 5000 only.

Solution. If $r' \neq m - 1$, then when the number $n - 1$ increases to n, remainder increases by 1 and quotient remains the same. In this case $r = r' + 1$, $q = q'$. Then, the equality $q + q' = r + r'$ can't hold since left hand side is even, while right hand side is odd.

So, $r' = m - 1$, $r = 0$, and, therefore, $q = q' + 1$. Hence, we have $2q - 1 = r'$, $r' = m - 1$.

a) We obtain $2q - 1 = m - 1 = 99$, so $m = 100$, $q = 50$. Then

$$n = qm + r = 50 \cdot 100 + 0 = 5000.$$

b) We obtain $2q - 1 = m - 1$, so $m = 2q$ and, finally,

$$2n = 2qm = 4q^2 = (2q)^2.$$

2. Nazar chose two real numbers a and b, and wrote two equations: $x^4 - 2b^3x + a^4 = 0$, and $x^4 - 2a^3x + b^4 = 0$. Prove that at least one of these equations has a real root.

Nazar Agakhanov

Solution. Since functions $f(x) = x^4 - 2b^3x + a^4$, $g(x) = x^4 - 2a^3x + b^4$ are polynomials with positive leading coefficient, it follows that $f(x) > 0$ and $g(x) > 0$ for all $x > c$, where $c \in \mathbb{R}$ is some constant. To prove that $f(x)$ has at least one root, we need to prove that for some x_0 the inequality $f(x_0) \leqslant 0$ holds. Then, the existence of the root follows from Intermediate Value Theorem. The same is true for $g(x)$.

The rest of the proof can be done it different ways:

1. Since, $f(b) = b^4 - 2b^4 + a^4 = a^4 - b^4$ and $g(a) = b^4 - a^4$, at least one of the numbers $f(b)$ and $g(a)$ is non-positive.

2. Since $f'(x) = 4x^3 - 2b^3$, we get that $f(x)$ decreases on interval $\left(-\infty; \frac{b}{\sqrt[3]{2}}\right]$ and increases on interval $\left[\frac{b}{\sqrt[3]{2}}; +\infty\right)$. So, the least value of f is at the point $\frac{b}{\sqrt[3]{2}}$,

$$f\left(\frac{b}{\sqrt[3]{2}}\right) = \frac{-3b^4}{2\sqrt[3]{2}} + a^4.$$

Similarly, the least value of $g(x)$ is $\frac{-3a^4}{2\sqrt[3]{2}} + b^4$.

The sum of these two values is equal to $k\left(a^4 + b^4\right)$, with $k = \frac{-3}{2\sqrt[3]{2}} + 1 < 0$ and, hence, non-positive. Therefore, at least one of the values is non-positive.

3. a) Determine if there exists a convex hexagon $ABCDEF$ with

$$\angle ABD + \angle AED > 180^\circ,$$
$$\angle BCE + \angle BFE > 180^\circ,$$
$$\angle CDF + \angle CAF > 180^\circ.$$

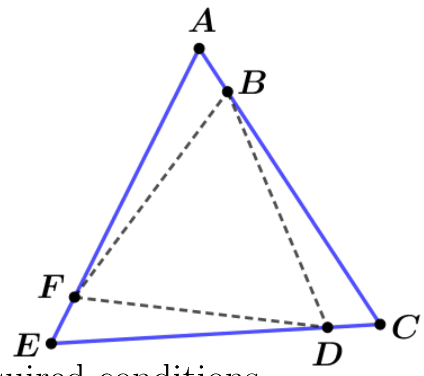
b) The same question, with additional condition, that diagonals AD, BE, and CF are concurrent.

Morteza Saghafian, Pavel Kozhevnikov

Answer. a) yes; b) no.

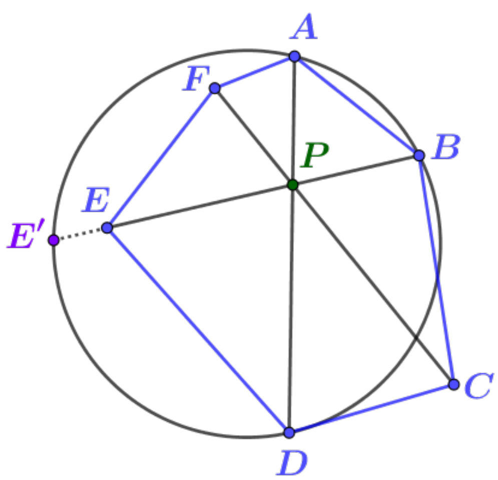
Solution. a) Let us present a construction.

On the sides AC, CE, EA of an equilateral triangle ACE let us mark points B, D, F «close» to A, C, E. Here the sum $\angle ABD + \angle AED$ is close to $180^\circ + 60^\circ = 240^\circ$. The same is true for the sums $\angle BCE + \angle BFE$ и $\angle CDF + \angle CAF$.



Now perform a «slight shift» of B, D, F to make them outside ACE. Now $ABCDEF$ satisfies the required conditions.

b) Assume the contrary, and let $ABCDEF$ be a required hexagon. Let P be a common point of AD, BE и CF.



Let ray PE intersect circle (ABD) at E'. Hence

$$\angle AE'D = 180^\circ - \angle ABD,$$

but, by condition of the problem,

$$180^\circ - \angle ABD < \angle AED.$$

From $\angle AE'D < \angle AED$ it follows that E' lies outside the segment PE (in other words, E lies inside circle (ABD)), hence $PE < PE'$.

Further, $PE \cdot PB < PE' \cdot PB = PD \cdot PA$. Thus, from the first inequality it follows that $PE \cdot PB < PD \cdot PA$. Similarly, obtain $PD \cdot PA < PF \cdot PC$, and $PF \cdot PC < PE \cdot PB$. Hence $PF \cdot PC < PF \cdot PC$, which is a contradiction.

4. Let n and k be integers, such that $n > k > 1$. Some pairs of n cities are joined by airlines so that for any k cities one of these k cities is joined with all other $k-1$ cities. Find the least possible number of airlines.

Vladimir Dol'nikov

Answer. $\dfrac{n(n-1)}{2} - \dfrac{(k-1)(k-2)}{2}$, for even k;

Solution. $\dfrac{n(n-1)}{2} - \max\left(\dfrac{(k-1)(k-2)}{2}; \left[\dfrac{n}{2}\right]\right)$, for odd k.

We will consider only the complement graph. We know that among any k vertices there is a vertex that is not connected to the others. Under this condition we need to find the largest possible number of edges.

Example. A complete graph on $k-1$ vertices (and all other vertices are isolated) obviously fits. If k is odd, any mathching is suitable, including the one with $\left[\frac{n}{2}\right]$ of edges.

Upper bound. We will call a subgraph *beautiful,* if each of its vertices has an edge inside it. The proof is by reductio ad absurdum. We assume that there are more edges in our graph than in the above examples, and in this case we find *beautiful* subgraph.

Below we will use a well-known fact: it is possible to remove a vertex from a connected graph (which has at least 2 vertices) so that after removal we get a connected graph again.

Upper bound for even *k*. Let the graph have more edges than the complete graph on $k-1$ vertices. Consider all components of our graph with at least one edge. Combine them into *beautiful* subgraph G, which contains all edges of our original graph. Since there are more of these edges than $\frac{(k-1)(k-2)}{2}$, there are at least k vertices. We will try to remove vertices from our subgraph one by one until there remains *beautiful* subgraph on k vertices. If the graph has a connectivity component with more than two vertices, then remove a vertex in it without loss of connectivity. If the graph has no such components, then all components consist of two vertices, choose $\frac{k}{2}$ from these components.

Upper bound for odd *k*. Note that there is a component K such that $|K| \geqslant 3$, otherwise no more than one edge incident to each vertex and there are at most $\left[\frac{n}{2}\right]$

total edges. If $|K| \geqslant k$, then dropping vertices out of it one by one and keeping connectivity, we will come to *beautiful* subgraph on k vertices.

Otherwise, if $|K| < k$, we follow the process as in the even case without touching K. Let the process stop at the *beautiful* subgraph Γ, which still has more than k vertices. Then Γ consists of K and several components of two vertices. In the case of $|Gamma| \geqslant k + 2$, let us continue removing components from two edges. If we come to the situation $|Gamma| = k+1$, then by removing one vertex from K, keeping the connectivity, we get a *beautiful* subgraph on k vertices.