XI Caucasus Mathematic Olympiad

Solutions book
Day 1

Caucasus
Mathematical
Olympiad

Кавказская
математическая
олимпиада

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March 13–18, 2026 year
Maykop
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Juniors. Day 1, March 14

1. *50 children are standing in a row. Determine if it is possible that among any* 5 *consecutive children the number of boys is less by* 1 *than the number of girls, and also among any* 7 *consecutive children the number of boys is less by* 1 *than the number of girls.*

I. A. Efremov

Answer: It could not.

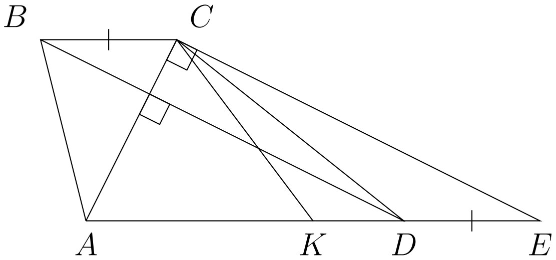
Solution. Suppose that such a situation could occur. It follows from the condition that among any 5 children standing in a row, there are exactly 2 boys, and among any 7 children standing in a row, there are exactly 3 boys. Let us divide all the children into 10 groups of 5 people standing in a row. Then the total number of boys is equal to $2 \cdot 10 = 20$, since in each group there are exactly 2 of them. On the other hand, the first 49 children can be divided into groups of 7 people standing in a row. Therefore, among them there are exactly $7 \cdot 3 = 21 > 20$ boys — a contradiction.

Solution 2. Again, suppose that such a situation could occur. Consider 35 children standing in a row. Divide them into 7 groups of 5 children, in each of which the children stand in a row. In each group there are exactly 2 boys. Therefore, the total number of boys among these 35 children is $7 \cdot 2 = 14$. On the other hand, these same children can be divided into 5 groups of 7 children standing in a row. Then the total number of boys among them is $5 \cdot 3 = 15 \neq 14$ — a contradiction.

2. *In a trapezoid $ABCD$ with $AD \parallel BC$ diagonals AC and BD are perpendicular. Let K be a point on the segment AD such that $AK + KC = AD + BC$. Prove that K is equidistant of A and C.*

L. A. Yemelyanov

Solution. On the extension of segment AD beyond point D, we mark point E such that $DE = BC$. Then quadrilateral $BCED$ is a parallelogram, since its sides BC and DE are equal and parallel. Hence $BD \parallel CE$. Therefore, in triangle ACE, we have $\angle ACE = 90^\circ$. Moreover, $KE = AD + DE - AK = AD + BC - AK = KC$, so triangle ACE is isosceles with base CE. Then $\angle CAK = 90^\circ - \angle AEC = 90^\circ - \angle ECK = \angle ACK$. Thus, triangle ACK is isosceles with base AC and $KA = KC$, as required.



3. *Find the smallest positive integer k such that the number 100^{100} can be represented as a product of 99 positive integers each of which is not greater than k.*

E. V. Bakaev

Answer: 128.

Solution. *Estimate.* Suppose that the number 100^{100} could be represented as a product of 99 natural numbers, each of which is less than 128. Since the number 100^{100} has no prime divisors other than 2 and 5, each of the 99 factors will be of the form $2^\alpha \cdot 5^\beta$.

Note that then $2^7 = 128 > 2^\alpha \cdot 5^\beta > 2^\alpha \cdot 2^{2\beta} = 2^{\alpha+2\beta}$, from which the inequality $\alpha + 2\beta < 7$ follows, i.e., $\alpha + 2\beta \leq 6$. Let us sum such inequalities over all 99 factors, using the fact that the total exponents of 2 and 5 in the product are equal to 200. We obtain $200 + 2 \cdot 200 \leq 99 \cdot 6$ — a contradiction.

Example. Let us prove that for $k = 128$ the required factorization exists. Suppose that among the factors, the number $128 = 2^7$ appears 6 times, the number $100 = 2^2 \cdot 5^2$ appears 79 times, and the number $125 = 5^3$ appears 14 times. Then the total exponent of 2 in the product is $6 \cdot 7 + 79 \cdot 2 = 200$, and the total exponent of 5 is $79 \cdot 2 + 14 \cdot 3 = 200$, as required.

4. *Pasha and Vova play a game on* 12×12 *checkered board. They take turns, Pasha starts. On his turn, Pasha can choose two empty cells adjacent by side and place a coin heads-up in each of these two cells. Vova, on his turn, can choose any two diagonally adjacent cells and make tails-up each coin already placed in these two cells. The game ends when Pasha can not perform next move. Find the greatest possible number of heads-up coins, that Pasha can achieve at the end of the game, regardless of Vova's actions?*

A. A. Solynin

Answer: $\frac{1}{4} \cdot 12^2 = 36$.

Solution. Note first that the game will always end, since Pasha can make at most $\frac{12^2}{2} = 72$ moves.

Let us show how Pasha can act so that at the end of the game there are at least $\frac{1}{4} \cdot 12^2$ coins lying heads up. Number all the rows with the numbers 1, 2, 3, …, 12 and split the rows with odd numbers into dominoes of two cells. We will make the first 36 moves into these dominoes. Note that with each reply move, Vova can reduce the number of coins lying heads up by at most 1. Therefore, after Vova's 36th move, at least $2 \cdot 36 - 36 = 36$ coins lying heads up will remain. After that, we split the rows with even numbers into dominoes and will move into them. Then after 36 moves, there will be at least one coin in each cell. Consequently, we will not be able to make the next move, and the game will end. Moreover, after each of our moves, exactly 2 coins appeared lying heads up, and after Vova's move, the number of such coins decreases by at most 2. Hence, at the end of the game, there will still be at least 36 coins lying heads up.

Now let us show how Vova can act so that at the end of the game there are certainly no more than 36 coins lying heads up. Color the cells of the board in a chessboard pattern and split the board into 36 2×2 squares. We will make the first 36 moves into a pair of black cells in one square of the partition. Moreover, on each next move, we will choose a pair of black cells, one of which Pasha moved into before us. If the game ends no later than after 36 pairs of moves, then no more than 36 coins lying heads up will remain. Otherwise, after our 36th move, there will be exactly 36 coins lying heads up on the board. Moreover, they will all lie on white cells.

Starting from the 37th move, we will make each of our moves into two white cells in one square of the partition. Let us prove that we can do this in such a way as to flip two coins lying heads up at once. Suppose that after our 36th move, k pairs of moves have passed, and now Pasha has made his next move. So far, we have made exactly k moves into white cells of different 2×2 squares. That is, there are exactly $36 - k$ squares of the partition left in which coins can lie heads up on the white cells. At the same time, in addition to the $2k$

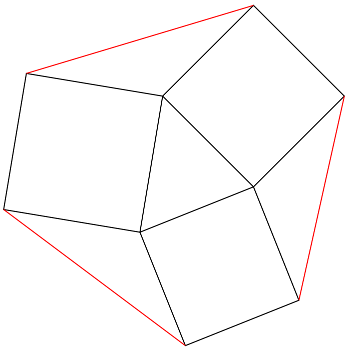
white cells we have previously "spoiled there are exactly $36 + k + 1 - 2k = 37 - k$ white cells whose coins lie heads up. Then, by the pigeonhole principle, in one of the remaining $36 - k$ 2×2 squares, there are two coins on white cells lying heads up. We will make our next move into this square.

Thus, after each subsequent pair of moves, the number of coins lying heads up does not increase. After the first 36 pairs of moves, there will be exactly 36 coins lying heads up. Therefore, at the end of the game, there will be no more than 36 coins lying heads up.

Seniors. Day 1, March 14

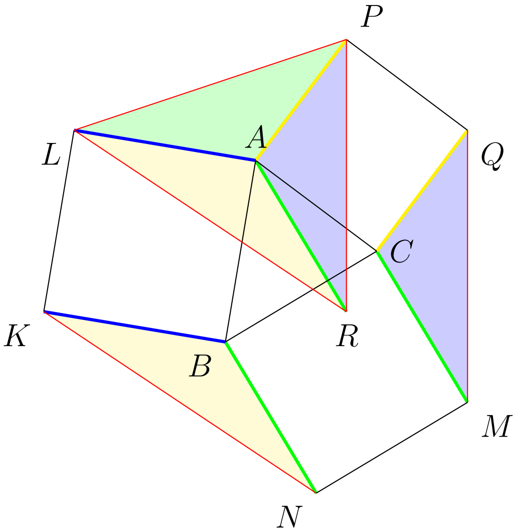
1. *On the sides of a triangle whose area equals 1 squares are constructed outward. The ends of the sides of the squares sharing a vertex of the triangle are connected by red segments (see figure).*

a) Prove that a triangle can be formed from three red segments; b) the area of this triangle equals 3.



L. A. Yemelyanov

Solution. Let the squares constructed on the sides of the given triangle ABC be the squares $ABKL$, $BCMN$, $CAPQ$ (see figure).



a) Let us perform a parallel translation of triangle KBN by the vector $\overrightarrow{BA} = \overrightarrow{KL}$, whereby it maps into triangle LAR. Similarly, translate triangle QCM by the vector $\overrightarrow{CA} = \overrightarrow{QP}$, whereby it maps into triangle PAR'. But $\overrightarrow{AR} = \overrightarrow{BN} = \overrightarrow{CM} = \overrightarrow{AR'}$, so the points R and R' coincide. Thus, a triangle PLR is formed, whose sides are equal to the red segments PL, $LR = KN$, and $RP = MQ$.

b) We see that the area of triangle PLR is the sum of the areas of triangles PAL, KBN, MCQ. Let us prove that the area of any of these is equal to the area of triangle ABC.

Indeed, $2S_{ABC} = AB \cdot AC \cdot \sin\angle BAC$, which is equal to $AL \cdot AP \cdot \sin\angle LAP = 2S_{PAL}$, since $AL = AB$, $AP = AC$, and $\angle BAC + \angle LAP = 360^\circ - \angle LAB - \angle PAC = 360^\circ - 90^\circ - 90^\circ = 180^\circ$.

2. *Peter arranged n positive integers in a circle. For each integer, Basil calculated the product of that number and the GCD of the two numbers following it clockwise. All the numbers Basil calculated turned out to be equal. Are the original numbers necessarily equal, if a) $n = 15$; б) $n = 16$?*

N. Kh. Agakhanov

Answer: a) no; b) yes.

Solution. a) A counterexample can be the cyclic sequence 2, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2, 1.

b) Let us number the numbers clockwise (here, numbers i and $i + n$ are considered the same). Consider a prime number p and let α_i be the exponent of this p in the factorization of the i-th number. Then by the condition we have $\alpha_1 + \min\{\alpha_2, \alpha_3\} = \alpha_2 + \min\{\alpha_3, \alpha_4\} = \ldots = \alpha_n + \min\{\alpha_1, \alpha_2\} = s$. Let $\alpha = \alpha_i$ be the maximum among all α_i, and let β be the minimum of α_{i+1} and α_{i+2}. Then $s = \alpha + \beta$, therefore the two numbers following β clockwise are equal to α (they are not less than α, but also not greater than α due to the choice of α). If $\alpha_j = \alpha_{j+1} = \alpha$, then $\alpha_{j+2} = \beta$, then $\alpha_{j+3} = \alpha_{j+4} = \alpha$, and so on. We obtain a cyclic sequence $\alpha, \alpha, \beta, \alpha, \alpha, \beta, \ldots$. If n is not divisible by 3, then after going around the circle we get $\alpha = \beta$.

Thus, all exponents α_i are equal. Repeating this argument for each prime number p, we obtain the equality of all the original numbers.

3. *Positive real numbers a, b, c satisfy*

$$\sqrt{a^2 + ab} + b + c = \sqrt{b^2 + bc} + c + a = \sqrt{c^2 + ca} + a + b.$$

Determine if it follows that $a = b = c$.

K. A. Sukhov

Answer: Yes.

Solution. Let b be the middle number, so that $a \geqslant b \geqslant c$ or $c \geqslant b \geqslant a$.

Consider the equality $\sqrt{a^2 + ab} + b + c = \sqrt{b^2 + bc} + c + a \iff \sqrt{a^2 + ab} - a = \sqrt{b^2 + bc} - b \iff \dfrac{ab}{\sqrt{a^2 + ab} + a} = \dfrac{bc}{\sqrt{b^2 + bc} + b} \iff \dfrac{\sqrt{a^2 + ab} + a}{a} = \dfrac{\sqrt{b^2 + bc} + b}{c} \iff \sqrt{1 + \frac{b}{a}} + 1 = \sqrt{\frac{b^2}{c^2} + \frac{b}{c}} + \frac{b}{c}$.

1) If $a \geqslant b \geqslant c$, we have $\sqrt{1 + \frac{b}{a}} + 1 \leqslant \sqrt{1 + 1} + 1$ and $\sqrt{\frac{b^2}{c^2} + \frac{b}{c}} + \frac{b}{c} \geqslant \sqrt{1 + 1} + 1$. Hence, both inequalities must become equalities, which is possible only when $a = b$ and $b = c$.

2) If $a \leqslant b \leqslant c$, then $\sqrt{1 + \frac{b}{a}} + 1 \geqslant \sqrt{1 + 1} + 1$ and $\sqrt{\frac{b^2}{c^2} + \frac{b}{c}} + \frac{b}{c} \leqslant \sqrt{1 + 1} + 1$. Again, both inequalities must become equalities, which is possible only when $a = b$ and $b = c$.

4. *We are given a sequence $a_1, a_2, \ldots, a_n$ of positive integers. At each step, all numbers change simultaneously as follows: the i-th number is replaced by the number of indices j such that $a_j = a_i$, plus i. (For example the sequence: $1, 1, 4, 1, 4, 2, 1, 3$ Becomes $5, 6, 5, 8, 7, 7, 11, 9$.) We then continue the process with the new sequence. Prove that after some time the sequence no longer changes.*

Morteza Saghafian

Solution. After each step (starting from the first), we obtain a sequence of n natural numbers, each of which does not exceed $2n$. The set of such sequences is finite, so some sequence P_1 will repeat twice. Let P_1, P_2, $\ldots$, P_k be the sequences that appear until the next occurrence of P_1 (so that $P_{k+1} = P_1$). Then P_1, P_2, $\ldots$, P_k will repeat periodically (each is obtained from the previous one in one step within the cycle).

For each of the sequences P_i, we find the maximum number of copies of any number occurring in it, and let m be the maximum of these quantities. Then in each of the sequences P_1, P_2, $\ldots$, P_k, every number exceeds its index by at most m (otherwise, in the previous sequence, the number with that index would have been repeated more than m times).

In particular, if $m = 1$, then each P_i coincides with the sequence $(2, 3, \ldots, n+1)$, which obviously will repeat. Suppose further that $m \geqslant 2$.

Among the sequences P_1, P_2, $\ldots$, P_k, consider the sequences with the largest number of copies (m) of some number x, and among them choose the one in which x is minimal. Without loss of generality, let the chosen sequence be P_t.

Consider the sequence $P_t = (b_1, b_2, \ldots, b_n)$, and also the sequence $P_{t-1} = (a_1, a_2, \ldots, a_n)$ (the indices t and $t-1$ are taken modulo k, so that P_t is obtained from P_{t-1} in one step).

Since in P_t there are m copies of the number x, let $b_{i_1} = \ldots = b_{i_m} = x$ for some indices $i_1 < \ldots < i_m$. Then $x = b_{i_m} \geqslant i_m + 1 \geqslant i_{m-1} + 2 \geqslant \ldots \geqslant i_1 + m$. We obtain $x = b_{i_1} \geqslant i_1 + m$. But as we noted, $b_{i_1} - i_1 \leqslant m$. Consequently, in the previous chain of inequalities, all must become equalities: $x = b_{i_m} = i_m + 1 = i_{m-1} + 2 \ldots = i_1 + m$, which equals b_{i_1}.

The last equality $x = i_1 + m$ means that a_{i_1} is repeated m times in the sequence P_{t-1}. Hence $a_{i_1} \geqslant x$ (according to the choice of x). Moreover, the strict inequality $a_{i_1} > x$ is impossible (otherwise $a_{i_1} - i_1 > m$).

Thus, $a_{i_1} = i_1 + m = x$, and x is repeated m times in the sequence P_{t-1}. But $a_j = x$ is impossible for $j < i_1$, otherwise $a_j - j > m$. Next, note that the numbers $a_{i_1}, a_{i_1+1}, \ldots, a_{i_1+m-1}$ are distinct, because if two of them were equal, then the corresponding numbers in the sequence P_t would be different (the same number of copies would be added to the index of the number), but as we recall, $b_{i_1}, b_{i_1+1}, \ldots, b_{i_1+m-1}$ are equal. Hence, among the numbers $a_{i_1+1}, \ldots, a_{i_1+m-1}$, there is no x. Finally, $a_j > x = a_{i_1} + m$ for $j \geqslant i_1 + m = x$. Thus, we understand that x appears exactly once in P_t. Contradiction.