XI Caucasus Mathematic Olympiad

Solutions book
Day 2

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Maykop
Adyghea

Juniors. Day 2, March 15

5. *Find the largest positive integer consisting of distinct non-zero digits such that for any two of its neighboring digits their product is a composite number.*

I. D. Beilin, E. D. Utkin, D. A. Belov

Answer: 987653241.

Solution. It is easy to verify that the number 987653241 satisfies the conditions of the problem. Note that the decimal representation of our number can have at most 9 digits. Suppose there exists a natural number greater than 987653241. Then its first 5 digits must be exactly 98765 in that order. Moreover, the number must contain all non-zero digits exactly once. In particular, the digit 1 must appear in its representation. However, it cannot be adjacent to 5, 2, or 3. Therefore, it must be at the end of the number, with the digit 4 before it. Then our number would be no greater than 987653241 — a contradiction to our assumption.

6. *Masha wrote down 2026 different positive real numbers in a notebook. She considers the pair of numbers a, b written in the notebook to be good if the number $\frac{2026-ab}{a+b}$ is equal to one of the remaining 2024 numbers. Determine if it is possible that among all the pairs of initial numbers there are exactly 2026 good pairs.*

I. A. Efremov

Answer: It could not.

Solution. Suppose that such a situation could occur. Consider a good pair of numbers a and b. Let $c = \frac{2026-ab}{a+b}$. By condition, the number c is in the notebook and is not equal to a and b. Multiplying the resulting equality by $a + b$, after moving ab to the left-hand side we obtain $ab + bc + ca = 2026$. Then $b(a + c) = 2026 - ac$, whence $b = \frac{2026-ac}{a+c}$. Hence, the pair of numbers a and c is also good. Similarly, the pair of numbers b and c is also good. Thus, we have associated with the good pair of numbers a, b two more good pairs b, c and c, a, where $c = \frac{2026-ab}{a+b}$. Moreover, if we apply the same rule, the good pair b, c will be associated with the pairs c, a and a, b. Similarly for the good pair c, a. Therefore, all good pairs of numbers can be partitioned into triples of this form. Hence, the total number of good pairs of numbers is divisible by 3. But 2026 is not divisible by 3 — a contradiction.

7. *In an 8×8 grid square all 81 grid points were marked. Igor connected some pairs of marked points with line segments to form an 81-gon. Prove that there exists a cell whose center lies on the boundary of this 81-gon.*

I. A. Efremov

Solution. Color the grid nodes in a chessboard pattern so that the number of black nodes is one more than the number of white nodes. Then, when traversing the 81-gon, there will be a side whose both ends are black.

Thus, we have a side connecting nodes A and B of the same color. Introduce coordinates $A(x_a, y_a)$, $B(x_b, y_b)$ (consistent with our grid). The fact that A and B are of the same color means that $x_a - x_b$ and $y_a - y_b$ have the same parity.

If both differences are even, then $x_a + x_b$ and $y_a + y_b$ are also even. Then the midpoint M of segment AB has coordinates $\left(\frac{x_a+x_b}{2}, \frac{y_a+y_b}{2}\right)$, and therefore is a grid node. This would mean that vertex M of our polygon lies on its side AB, which is impossible.

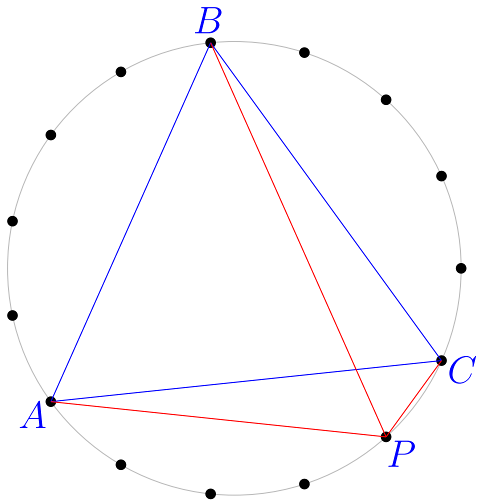
Therefore, both differences $x_a - x_b$ and $y_a - y_b$ are odd. Then the sums $x_a + x_b$ and $y_a + y_b$ are also odd. In this case, the midpoint M of segment AB has coordinates of the form $\left(k + \frac{1}{2}, \ell + \frac{1}{2}\right)$ for some integers k and ℓ. Hence, M coincides with the center of one of the grid squares and lies on the side of the polygon, as required.

8. *A regular* 111*-gon with side length 1 is given. Prove that there are 2 diagonals whose lengths differ by 1.*

E. V. Bakaev

Solution. We use the following well-known *statement*. Let ABC be an equilateral triangle, and let point P be chosen on the minor arc CA of the circumcircle (ABC). Then $PB = PA + PC$.

Proof of the statement. On the extension of segment PA beyond point A, mark point D such that $AD = CP$. From the cyclic quadrilateral $ABCP$, we obtain $\angle DAB = 180^\circ - \angle PAB = \angle PCB$. Then triangles DAB and PCB are congruent by two sides and the included angle: $DA = PC$, $AB = CB$, and $\angle DAB = \angle PCB$. Hence, $DB = BP$, and triangle DBP is isosceles with base DP. But $\angle DPB = \angle APB = \angle ACB = 60^\circ$. Therefore, triangle DBP is equilateral and $BP = PD = PA + PC$.



Now return to the solution of the problem. Let $A_1A_2 \dots A_{3k}$ be our regular 111-gon ($k = 37$). Then $A = A_k$, $B = A_{2k}$, $C = A_{3k}$ are the vertices of an equilateral triangle, and point $P = A_1$ lies on the minor arc CA of its circumcircle. Then, according to the statement, $PB = PA + PC$. But $PC = 1$, since PC is a side of our 111-gon. Hence $PB - PA = 1$, therefore PB and PA are the required diagonals.

Seniors. Day 2, March 15

5. *Determine if there exist non-constant linear functions $f(x)$ and $g(x)$ such that the equation $f(x)g(x+1) + f(x+1)g(x) = 0$ has no real roots.*

I. A. Efremov

Answer: They do not exist.

Solution. Assume the contrary, and let $f(x) = ax + c$, $g(x) = bx + d$, with a, $b \neq 0$. Our equation $f(x)g(x+1) + f(x+1)g(x) = 0$ takes the form $(ax+c)(b(x+1)+d) + (a(x+1)+c)(bx+d) = 0$.

Set $p = c/a$, $q = d/b$. Dividing by ab, we obtain a simpler form $F(x) = 0$, where $F(x) = (x+p)(x+1+q) + (x+p+1)(x+q)$.

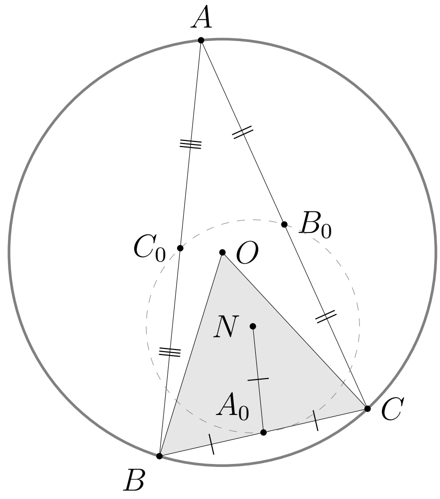
Note that $F(-p) = 0 + 1 \cdot (-p + q) = q - p$ and similarly $F(-q) = p - q$. If $p = q$, then $-p$ is a root of the equation $F(x) = 0$. Otherwise, we see that the (quadratic) function F takes values of opposite signs, and therefore has a root.

Remark. Of course, the existence of roots can be proved directly — by noting that the discriminant of the quadratic trinomial $(x+p)(x+1+q)+(x+p+1)(x+q)$ is nonnegative (in fact, $D/4 = (p - q)^2 + 1$).

6. *Let ABC be a triangle with $\angle BAC = 30°$. Let N be the center of the circle passing through the midpoints of AB, BC, and CA. Prove that $\angle BNC = 90°$.*

L. A. Yemelyanov

Solution. Let C_0, A_0, B_0 be the midpoints of segments AB, BC, CA respectively. It suffices to understand that $NA_0 = BC/2$, i.e., that in triangle BNC, the median drawn from N is equal to half the opposite side.



Let the circumcircle of (ABC) have center O and radius R. Since $\angle BOC = 2\angle BAC = 60°$ and $OB = OC = R$, triangle BOC is equilateral, hence $BC = R$.

On the other hand, NA_0 is the radius of the circumcircle of triangle $A_0B_0C_0$, which is similar to triangle ABC with ratio $1/2$, so $NA_0 = R/2$.

Thus, $NA_0 = R/2 = BC/2$, which is what we wanted to prove.

Remark. The point N is known as the center of the nine-point circle (which passes through the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments joining the orthocenter to the vertices). It is known that N is the midpoint of segment OH (where H is the orthocenter, and O is, as before, the circumcenter). This description can be used for another solution, a sketch of which we provide below.

Fix points B and C, and allow point A to move along a fixed arc (so that $\angle BAC = \alpha$ is constant). It is not difficult to understand that then the orthocenter H moves along a fixed circle Ω — the circle symmetric to (ABC) with respect to BC. Under a homothety with center O and ratio $1/2$, point H maps to N. Consequently, N lies on the circle ω, which is the image of Ω under a homothety with center O and ratio $1/2$. One can verify that in the case $\alpha = 30°$ (as well as $\alpha = 150°$), the resulting circle ω coincides with the circle constructed on BC as a diameter. (One can also show that for other values of α, the circle ω does not pass through B and C, so the angle BNC is not constant).

7. *Let positive integers $a_1 < a_2 < \ldots < a_{120}$ form an arithmetic progression. Find the greatest possible number of pairs $1 \leqslant i < j \leqslant 120$ such that a_j is divisible by a_i.*

P. A. Kozhevnikov

Answer: $482 = \sum_{i=2}^{120} \lfloor \frac{120}{i} \rfloor$.

Solution. Set $n = 120$.

Estimate. For a given i, estimate the number of indices $j \leqslant n$ such that a_j is divisible by a_i. Let $i = j_1 < j_2 < \ldots < j_s$ be all such indices. Note that $a_i > (i-1)d$. Since $a_{j_{t+1}} - a_{j_t} = (j_{t+1} - j_t)d$ is divisible by a_i, we obtain $(j_{t+1} - j_t)d \geqslant a_i > (i-1)d$, whence (dividing by d) $j_{t+1} - j_t > i - 1$, and therefore $j_{t+1} - j_t \geqslant i$. From this we successively obtain the inequalities $j_1 \geqslant i$, $j_2 \geqslant 2i$, $\ldots$, $j_s \geqslant si$. Since $j_s \leqslant n$, we have $si \leqslant n$, hence $s \leqslant \lfloor \frac{n}{i} \rfloor$. Summing these estimates for all $i = 1, 2, \ldots, n$ and taking into account that we have counted n pairs of coinciding indices (the pairs i, i). Finally, the desired number of index pairs does not exceed $\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor - n = \sum_{i=2}^{n} \lfloor \frac{n}{i} \rfloor$.

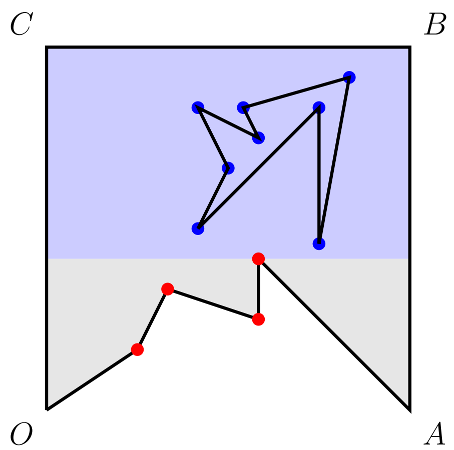
Example. Consider the numbers 1, 2, 3, $\ldots$, n. It is easy to see that in this example, the number of required pairs is indeed $\sum_{i=2}^{n} \lfloor \frac{n}{i} \rfloor$ (since all the inequalities used in the estimate become equalities).

Remark. One could try to estimate the number of desired pairs $i < j$ with a fixed quotient $k = a_j/a_i$. However (without additional considerations), this does not always yield an exact estimate. For example, for $n = 120$ in the optimal example $(1, 2, \ldots, 120)$ the quotient $k = 11$ occurs for 10 pairs, while, say, in the progression $9, 19, 29, \ldots, 1199$ the quotient 11 occurs in 11 pairs.

8. *In a plane, four vertices of a square and 2026 points inside this square are marked. No three of the marked points are collinear. Prove that there exist two 1015-gons with vertices at the marked points, one of which lies strictly inside the other.*

S. G. Volchyonkov

Solution. Introduce coordinates so that our square K has vertices $O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$. Let A_1, $\ldots$, A_{2026} be the remaining 2026 marked points, ordered by increasing ordinate. (Since no three marked points are collinear, ordinates can be equal for at most two marked points.) Color the points A_1, $\ldots$, A_{1011} red, and the remaining 1015 marked points A_{1012}, $\ldots$, A_{2026} blue.



Rename the red points as B_1, $\ldots$, B_{1011} in order of increasing abscissa. Then let the boundary of the outer polygon P_1 be the non-self-intersecting closed polygonal line $ABCOB_1B_2\ldots B_{1011}A$. This polygonal line is non-self-intersecting, since the segments AB,

BC, CO lie on the convex hull, and the vectors $\overrightarrow{OB_1}$, $\overrightarrow{B_1B_2}$, $\ldots$, $\overrightarrow{B_{1010}B_{1011}}$, $\overrightarrow{B_{1011}A}$ have non-negative abscissas (they are directed to the right), and among them there are no two consecutive vectors parallel to the Oy axis.

All blue points lie in the rectangle Π, formed by the lines AB, BC, CO and the horizontal line ℓ passing through the topmost red point (the point A_{1011}). At the same time, at most one blue point lies on the boundary of this rectangle (possibly only A_{1012}), and the entire rectangle Π belongs to the polygon P_1, hence the convex hull of all blue points lies strictly inside P_1.

It remains to connect all blue points with a closed non-self-intersecting polygonal line (which will be the boundary of the desired polygon P_2). As is known, this can be done. For example, such a polygonal line can be the closed polygonal line of minimal length among all closed polygonal lines with vertices at all blue points. (If this polygonal line had intersecting segments XY and ZT, then one could replace them with one of the pairs XZ, YT or XT, YZ and obtain a closed polygonal line of smaller length.)

Remark. Using a similar method, one can prove the following statement, which generalizes the assertion of the problem. Suppose n points in general position are marked on the plane, and their convex hull is an m-gon. Then for any $k \geqslant m$, there exist a k-gon and an $(n-k)$-gon with vertices at the marked points such that the latter lies inside the former.